Exercise 2.29

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Task: Evaluate the sum $\sum_{k=1}^{n}(-1)^k\frac{k}{4k^2-1}$

Solution: We have a sequence of *A* (1) and we're looking for a sum S_n (2).

$$a_n=(-1)^n\frac{n}{4n^2-1} \tag{1}$$

$$S_n=\sum_{k=1}^{n}a_k \tag{2}$$

First of all we can rewrite a_n as:

$$a_n=(-1)^n\frac{n}{(2n-1)(2n+1)} \tag{3}$$

Let's divide both the numerator and denominator of (3) with *(2n-1):*

$$\begin{aligned} a_n &= (-1)^n\frac{\frac{n}{2n-1}}{2n+1} \\ &= \frac{(-1)^n}{2}\frac{\frac{2n-1}{2n-1}+\frac{1}{2n-1}}{2n+1} \\ &= \frac{(-1)^n}{2}\left(\frac{1}{2n+1}+\frac{1}{(2n-1)(2n+1)}\right) \end{aligned} \tag{4}$$

Then let's repeat the same process with *(2n+1)*:

$$\begin{aligned} a_n &= (-1)^n\frac{\frac{n}{2n+1}}{2n-1} \\ &= \frac{(-1)^n}{2}\frac{\frac{2n+1}{2n+1}-\frac{1}{2n+1}}{2n-1} \\ &= \frac{(-1)^n}{2}\left(\frac{1}{2n-1}-\frac{1}{(2n-1)(2n+1)}\right) \end{aligned} \tag{5}$$

Adding (4) and (5) together we get:

$$\begin{aligned} 2a_n &= \frac{(-1)^n}{2}\left(\frac{1}{2n+1}+\frac{1}{(2n-1)(2n+1)}+\frac{1}{2n-1}-\frac{1}{(2n-1)(2n+1)}\right) \\ a_n &= \frac{(-1)^n}{4}\left(\frac{1}{2n+1}+\frac{1}{2n-1}\right) \end{aligned} \tag{6}$$

Let's bring in new sequences *B* and *C* such that:

$$\begin{aligned} b_n &= (-1)^n \frac{1}{2n-1} \\ c_n &= (-1)^n \frac{1}{2n+1} \end{aligned} \tag{7}$$

We can represent the original sequence *A* thru *B* and *C*:

$$a_n=\frac{1}{4}\left((-1)^n\frac{1}{2n-1}+(-1)^n\frac{1}{2n+1}\right)=\frac{1}{4}(b_n+c_n) \tag{8}$$

The sums of sequences *B* and *C* would be:

$$\begin{aligned} T_n &= \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(-1)^k\frac{1}{2k-1} \\ U_n &= \sum_{k=1}^{n} c_k = \sum_{k=1}^{n}(-1)^k\frac{1}{2k+1} \end{aligned} \tag{9}$$

From (8) and (9) we can represent the original S_n as:

$$S_n=\sum_{k=1}^{n} a_k=\sum_{k=1}^{n}\frac{1}{4}(b_k+c_k)=\frac{1}{4}\left(\sum_{k=1}^{n} b_k+\sum_{k=1}^{n} c_k\right)=\frac{1}{4}(T_n+U_n) \tag{10}$$

Using something vaguely similar to perturbation method we can rewrite T_n and U_n as:

$$T_n=b_1+\sum_{k=2}^{n} b_k=-1\frac{1}{2\cdot1-1}+\sum_{k=2}^{n}(-1)^k\frac{1}{2k-1} \tag{11}$$

$$U_n=\sum_{1}^{n-1} c_k+c_n=\sum_{k=1}^{n-1}(-1)^k\frac{1}{2k+1}+(-1)^n\frac{1}{2n+1} \tag{12}$$

When we replace index *k* with *m* so that *m = k - 1* in formula (11) we get:

$$T_n=-1+\sum_{m=1}^{n-1}(-1)^{m+1}\frac{1}{2(m+1)-1}=-1+(-1)\sum_{m=1}^{n-1}(-1)^m\frac{1}{2m+1} \tag{13}$$

According to (10) adding formulas (13) and (12) together should give us quadruple of S_n:

$$4S_n=-1-\sum_{m=1}^{n-1}(-1)^m\frac{1}{2m+1}+\sum_{k=1}^{n-1}(-1)^k\frac{1}{2k+1}+(-1)^n\frac{1}{2n+1} \tag{14}$$

The names of the indexes don't matter therefore the two sums in (14) are equal and S_n is:

$$S_n=\frac{1}{4}\left(-1+(-1)^n\frac{1}{2n+1}\right) \tag{15}$$