Exercise 5.38

Hendrik Maarand

Problem: Show that all nonnegative integers n can be represented uniquely in the form $n = \binom{a}{1} + \binom{b}{2} + \binom{c}{3}$ where a, b, and c are integers with $0 \leq a < b < c$.

Existence: Let's first show that all nonnegative integers can indeed be represented this way.

We need to find a c such that $\binom{c}{3} \leq n$. This can be done by finding the maximum c from the sequence $\langle 0, 1, \ldots \rangle$ that satisfies the criteria. The maximum exists since $\binom{c}{3}$ is a strictly increasing function of c in the range $[3, +\infty)$ and 0 everywhere else. Similar arguments apply for $\binom{b}{2}$ and $\binom{a}{1}$.

$$
\begin{aligned}
c' &= max\ c.\ \binom{c}{3} \leq n & n' &= n - \binom{c'}{3} \\
b' &= max\ b.\ \binom{b}{2} \leq n' & n'' &= n' - \binom{b'}{2} \\
a' &= max\ a.\ \binom{a}{1} \leq n'' & n''' &= n'' - \binom{a'}{1} \\
&= n'' & &= n'' - n'' = 0
\end{aligned}
$$

For this choice of a', b', and c' to be a valid representation, we need to show that $0 \leq a' < b' < c'$.

By our choice of c' we know that

$$n < \binom{c'+1}{3}$$

which is equivalent to

$$n - \binom{c'}{3} < \binom{c'+1}{3} - \binom{c'}{3}$$

which after applying the definition of n' and the binomial identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ gives

$$n' < \binom{c'}{2}$$

Since $\binom{b'}{2} \leq n' < \binom{c'}{2}$ it must be the case that $b' < c'$. We can apply similar reasoning to get $a' < b'$. Since a' is chosen from $\langle 0, 1, \ldots \rangle$ we have that $0 \leq a' < b' < c'$ and thus $\binom{a'}{1} + \binom{b'}{2} + \binom{c'}{3}$ is a valid representation of n.

Uniqueness: To show that the representation is unique we need to show that we cannot find another triple (a'', b'', c'') that is also a valid representation of n.

We must only consider c'' that are less than c' since otherwise we would overflow n. Let's consider the case where we pick c'' so that $b' < c''$. This means that we are left with extra $\binom{c'}{3} - \binom{c''}{3}$ that we need to redistribute between the other two binomials. Let's say that we wish to decrease c' by m, that is $c'' = c' - m$. We can do this by applying the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ m times.

$$
\begin{align}
n &= \binom{a'}{1} + \binom{b'}{2} + \binom{c'}{3} \tag{1}\\
&= \binom{a'}{1} + \binom{b'}{2} + \binom{c'-1}{2} + \binom{c'-1}{3} \tag{2}\\
&= \binom{a'}{1} + \binom{b'}{2} + \binom{c'-1}{2} + \ \dots \ + \binom{c'-m}{2} + \binom{c'-m}{3} \tag{3}\\
&= \binom{a'}{1} + \binom{b'}{2} + \binom{c'-1}{2} + \ \dots \ + \binom{c'-m+1}{2} \tag{4}\\
&+ \binom{c'-m-1}{1} + \binom{c'-m-1}{2} + \binom{c'-m}{3} \nonumber
\end{align}
$$

In equation 3 we are in a situation where $\binom{c'-m}{3}$ is the term $\binom{c''}{3}$ we want and now we need to deal with $\binom{b'}{2} + \binom{c'-1}{2} + \ \dots \ + \binom{c'-m}{2}$ to find b''. We can observe that b'' can be at most $c' - m - 1$ and we can extract $\binom{c'-m-1}{2}$ by applying the binomial identity to $\binom{c'-m}{2}$ arriving at equation 4. Now we need to deal with $\binom{b'}{2} + \binom{c'-1}{2} + \ \dots \ + \binom{c'-m+1}{2} + \binom{c'-m-1}{1}$ to find a''. We can start by adding $\binom{a'}{1}$ and $\binom{c'-m-1}{1}$ to get $\binom{a'+c'-m-1}{1}$. For this to be valid, $a' + c' - m - 1 < c' - m - 1$ must hold, which implies that $a' < 0$ which cannot be since a' is nonnegative.

We also need to consider the case where we choose c'' such that $c'' \leq b'$ which means that we also need to decrease b'. By applying similar reasoning, if we try to decrease b' by p, we can see that it is not possible.

$$
\begin{aligned}
n' &= \binom{a'}{1} + \binom{b'}{2}\\
&= \binom{a'}{1} + \binom{b'-1}{1} + \ \dots \ + \binom{b'-p}{1} + \binom{b'-p}{2}
\end{aligned}
$$

Here we can add $\binom{a'}{1}$ and $\binom{b'-p}{1}$ to get $\binom{a'+b'-p}{1}$ in which case it must be that $a' + b' - p < b' - p$ which implies that $a' < 0$ which again is not possible. Therefore, it must be that $\binom{a'}{1} + \binom{b'}{2} + \binom{c'}{3}$ is a unique representation of n.