# Two Sums of Stirling Subset Numbers Homework for ITT9131 Concrete Mathematics 

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## 1 The Problem

Exercise 6.32 in the course textbook: we obtained the formulas

$$
\sum_{k \leq m}\binom{n+k}{k}=\binom{n+m+1}{m}
$$

and

$$
\sum_{0 \leq k \leq m}\binom{k}{n}=\binom{m+1}{n+1}
$$

by unfolding the recurrence

$$
\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}
$$

in two ways. What identities appear when the analogous recurrence

$$
\left\{\begin{array}{l}
n \\
k
\end{array}\right\}=k\left\{\begin{array}{c}
n-1 \\
k
\end{array}\right\}+\left\{\begin{array}{l}
n-1 \\
k-1
\end{array}\right\}
$$

is unwound?

## 2 Unwinding to the Right

With such a clear hint in the problem statement, it's natural to start from just checking what we get by unwinding (assuming $m, n \geq 0$ ):

$$
\begin{aligned}
\left\{\begin{array}{c}
n+m+1 \\
m
\end{array}\right\} & =m\left\{\begin{array}{c}
n+m \\
m
\end{array}\right\}+\left\{\begin{array}{c}
n+m \\
m-1
\end{array}\right\} \\
& =m\left\{\begin{array}{c}
n+m \\
m
\end{array}\right\}+(m-1)\left\{\begin{array}{c}
n+m-1 \\
m-1
\end{array}\right\}+\left\{\begin{array}{c}
n+m-1 \\
m-2
\end{array}\right\} \\
& \ldots \\
& =m\left\{\begin{array}{c}
n+m \\
m
\end{array}\right\}+\cdots+2\left\{\begin{array}{c}
n+2 \\
2
\end{array}\right\}+1\left\{\begin{array}{c}
n+1 \\
1
\end{array}\right\}+\left\{\begin{array}{c}
n+1 \\
0
\end{array}\right\} .
\end{aligned}
$$

Noting that $\left\{\begin{array}{c}n+1 \\ 0\end{array}\right\}=0$ for $n \geq 0$ and assuming the original combinatorial interpretation of $\left\{\begin{array}{l}n \\ m\end{array}\right\}$ (where $\left\{\begin{array}{l}n \\ m\end{array}\right\}=0$ for all $m<0$ irrespective of $n$ ), we get the hypothesis

$$
\sum_{k \leq m} k\left\{\begin{array}{c}
n+k \\
k
\end{array}\right\}=\left\{\begin{array}{c}
n+m+1 \\
m
\end{array}\right\}
$$

which we can prove by induction:
Base: For $m=0$, we have

$$
\sum_{k \leq 0} k\left\{\begin{array}{c}
n+k \\
k
\end{array}\right\}=0=\left\{\begin{array}{c}
n+1 \\
0
\end{array}\right\}
$$

as $\left\{\begin{array}{c}n+k \\ k\end{array}\right\}=0$ for all $k<0$ and $k\left\{\begin{array}{c}n+k \\ k\end{array}\right\}=0$ for $k=0$ on the left side, and also $\left\{\begin{array}{c}n+1 \\ 0\end{array}\right\}=0$ for $n \geq 0$ on the right side, so indeed both sides vanish.

Step: Assume $\sum_{k \leq m} k\left\{\begin{array}{c}n+k \\ k\end{array}\right\}=\left\{\begin{array}{c}n+m+1 \\ m\end{array}\right\}$. Then

$$
\begin{aligned}
\sum_{k \leq m+1} k\left\{\begin{array}{c}
n+k \\
k
\end{array}\right\} & =\sum_{k \leq m} k\left\{\begin{array}{c}
n+k \\
k
\end{array}\right\}+(m+1)\left\{\begin{array}{c}
n+m+1 \\
m+1
\end{array}\right\} \\
& =\left\{\begin{array}{c}
n+m+1 \\
m
\end{array}\right\}+(m+1)\left\{\begin{array}{c}
n+m+1 \\
m+1
\end{array}\right\} \\
& =\left\{\begin{array}{c}
n+m+2 \\
m+1
\end{array}\right\},
\end{aligned}
$$

which again matches, and so we have proven our hypothesis (for $n, m \geq 0$, and assuming the combinatorial interpretation of $\left\{\begin{array}{l}n \\ m\end{array}\right\}$ ).

## 3 Unwinding to the Left

Like in the previous case, we start by following the hint in the problem statement and unwinding (assuming $m, n \geq 0$ again):

$$
\begin{aligned}
\left\{\begin{array}{c}
m+1 \\
n+1
\end{array}\right\} & =(n+1)\left\{\begin{array}{c}
m \\
n+1
\end{array}\right\}+\left\{\begin{array}{c}
m \\
n
\end{array}\right\} \\
& =(n+1)\left((n+1)\left\{\begin{array}{c}
m-1 \\
n+1
\end{array}\right\}+\left\{\begin{array}{c}
m-1 \\
n
\end{array}\right\}\right)+\left\{\begin{array}{c}
m \\
n
\end{array}\right\} \\
& =(n+1)^{2}\left\{\begin{array}{c}
m-1 \\
n+1
\end{array}\right\}+(n+1)\left\{\begin{array}{c}
m-1 \\
n
\end{array}\right\}+\left\{\begin{array}{c}
m \\
n
\end{array}\right\} \\
& \ldots \\
& =(n+1)^{m+1}\left\{\begin{array}{c}
0 \\
n+1
\end{array}\right\}+(n+1)^{m}\left\{\begin{array}{l}
0 \\
n
\end{array}\right\}+\cdots+(n+1)\left\{\begin{array}{c}
m-1 \\
n
\end{array}\right\}+\left\{\begin{array}{c}
m \\
n
\end{array}\right\} .
\end{aligned}
$$

Noting that $\left\{\begin{array}{c}0 \\ n+1\end{array}\right\}=0$ for $n \geq 0$, we get the hypothesis

$$
\sum_{0 \leq k \leq m}(n+1)^{m-k}\left\{\begin{array}{l}
k \\
n
\end{array}\right\}=\left\{\begin{array}{l}
m+1 \\
n+1
\end{array}\right\}
$$

which we can prove by induction:
Base: For $m=0$, we have

$$
\sum_{0 \leq k \leq 0}(n+1)^{-k}\left\{\begin{array}{l}
k \\
n
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
n
\end{array}\right\}=[n=0]=\left\{\begin{array}{c}
1 \\
n+1
\end{array}\right\}
$$

as it should be.
Step: Assume $\sum_{0 \leq k \leq m}(n+1)^{m-k}\left\{\begin{array}{l}k \\ n\end{array}\right\}=\left\{\begin{array}{c}m+1 \\ n+1\end{array}\right\}$. Then

$$
\begin{aligned}
\sum_{0 \leq k \leq m+1}(n+1)^{(m+1)-k}\left\{\begin{array}{l}
k \\
n
\end{array}\right\} & =\sum_{0 \leq k \leq m}(n+1)^{(m-k)+1}\left\{\begin{array}{l}
k \\
n
\end{array}\right\}+(n+1)^{0}\left\{\begin{array}{c}
m+1 \\
n
\end{array}\right\} \\
& =(n+1) \sum_{0 \leq k \leq m}(n+1)^{m-k}\left\{\begin{array}{l}
k \\
n
\end{array}\right\}+\left\{\begin{array}{c}
m+1 \\
n
\end{array}\right\} \\
& =(n+1)\left\{\begin{array}{c}
m+1 \\
n+1
\end{array}\right\}+\left\{\begin{array}{c}
m+1 \\
n
\end{array}\right\} \\
& =\left\{\begin{array}{l}
m+2 \\
n+2
\end{array}\right\},
\end{aligned}
$$

which again matches, and so we have proven our hypothesis (for $n, m \geq 0$ ).

