Binomial Coefficients and Generating Functions

ITT9131 Konkreetne Matemaatika

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Binomial coefficients

Definition

Let r be a real number and k an integer. The binomial coefficient "r choose k" is the real number

$$\binom{r}{k} = \begin{cases} \dfrac{r\cdot(r-1)\cdots(r-k+1)}{k!} = \dfrac{r^{\underline{k}}}{k!} & \text{if } k \geqslant 0\,, \\ 0 & \text{if } k < 0\,. \end{cases}$$

If $r = n$ is a natural number

In this case,

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!}$$

is the number of ways we can choose k elements from a set of n elements, in any order.

Consistently with this interpretation,

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leqslant k \leqslant n\,, \\ 0 & \text{if } k > n\,. \end{cases}$$

Binomial identities cheat sheet

- $\binom{r}{0} = 1$ for every r real.
- $\binom{r}{1} = r$ for every r real.
- $\binom{n}{n} = [n \geqslant 0]$ for every n integer.
- $\binom{0}{k} = [k = 0]$ for every k integer.
- $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$ for every r real and $k \neq 0$ integer.
- $k\binom{r}{k} = r\binom{r-1}{k-1}$ for every r real and k integer (also 0).
- $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ for every r real and k integer.
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for every $n \geqslant 0$ integer.
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n = 0]$ for every $n \geqslant 0$ integer.

Unfolding

Theorem

For every r real and n integer,

$$\sum_{k \leqslant n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

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Proof

If $n < 0$ both sides vanish; if $n = 0$ then both sides equal 1.
Otherwise, the identity:

$$\binom{r+n+1}{n} = \binom{r+n}{n} + \binom{r+n}{n-1}$$

shows that $U_n = \binom{r+n+1}{n}$, $n \geqslant 0$, is the unique solution to the recurrence:

$$\begin{array}{rcl} U_0 & = & 1\,, \\ U_n & = & U_{n-1} + \binom{r+n}{n} \quad \forall n \geqslant 1\,. \end{array}$$

But such recurrence is clearly satisfied by $U_n = \sum_{k=0}^{n} \binom{r+k}{k}$.

Summation on the upper index

Theorem

For every $m, n \geqslant 0$ integers,

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

Warning: for $m = -1$ and $n = 0$, the left-hand side is 0, but the right-hand side is 1.

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Proof

The identity:

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shows that $U_n = \binom{n+1}{m+1}$ is the unique solution to the recurrence:

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But such recurrence is clearly satisfied by $U_n = \sum_{k=0}^{n} \binom{k}{m}$.

Going halves

Theorem

For every r real and k nonnegative integer,

$$\binom{r}{k}\binom{r-1/2}{k} = 2^{-2k}\binom{2r}{2k}\binom{2k}{k}$$

Indeed,

$$\begin{aligned} r^{\underline{k}}(r-1/2)^{\underline{k}} &= r(r-1/2)(r-1)(r-3/2)\cdots(r-k+1)(r-k+1/2) \\ &= \frac{2r(2r-1)\cdots(2r-2k+2)(2r-2k+1)}{2^{2k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}} \; ; \end{aligned}$$

so that:

$$\frac{r^{\underline{k}}}{k!}\frac{(r-1/2)^{\underline{k}}}{k!} = 2^{-2k}\frac{(2r)^{\underline{2k}}}{(2k)!}\frac{(2k)!}{(k!)^2}$$

Higher order differences

We define the nth order difference by induction on $n \geqslant 0$:

$$\Delta^0 f = f \text{ and } \Delta^{n+1} f = \Delta(\Delta^n f)$$

For instance, $\Delta^2 f(x) = \Delta(f(x+1) - f(x)) = f(x+2) - 2f(x+1) + f(x)$.

Theorem

For every $n \geqslant 0$,

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k)$$

Indeed, if E is the *shift* and I is the identity, then $\Delta = E - I$: as E and I commute, it is licit to perform

$$\Delta^n = (E-I)^n = \sum_{k=0}^{n} E^k (-I)^{n-k}$$

Warning: this is only possible because $E \circ I = I \circ E$!

The binomial inversion formula

Theorem (Binomial inversion formula)

Let f and g be two complex- functions defined on $\mathbb{N}$. The following are equivalent:

1. For every $n \geqslant 0$, $g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k)$.
2. For every $n \geqslant 0$, $f(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(k)$.

Note that the role of f and g is symmetrical.

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Note that the role of f and g is symmetrical.

Proof

If $g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k)$ for every $n \geqslant 0$, then for every $n \geqslant 0$ also:

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-1)^k g(k) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{m=0}^{k} \binom{k}{m} (-1)^m f(m) \\
&= \\
&= \sum_{0 \leqslant m \leqslant k \leqslant m} \frac{n^{\underline{m}} (n-m)^{\underline{k-m}}}{k!} \frac{k!}{m!(k-m)!} (-1)^{k-m} f(m) \\
&= \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{k=m}^{n} (-1)^{k-m} \binom{n-m}{k-m} \right) f(m) \\
&= \sum_{m=0}^{n} \binom{n}{m} [m = n] \, f(m) = f(n) .
\end{aligned}
$$

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Generating functions

Let

$$\langle g_0, g_1, g_2, \ldots, g_n, \ldots \rangle$$

with $n \in \mathbb{N}$ be a sequence of generic term g_n.
The generating function of the sequence is the power series

$$G(z) = \sum_{n \geqslant 0} g_n z^n = \lim_{N \to \infty} \sum_{n=0}^{N} g_n z^n ,$$

which is defined for z in a suitable neighborhood of the origin of the complex plane.

Three basic principles

1. Every complex-valued function of a complex variable having complex derivative in a neighborhood of the origin of the complex plane is the generating function of some sequence.
2. Every sequence $\{g_n\}_{n\geqslant 0}$ of complex numbers such that

$$M = \limsup_n \sqrt[n]{|g_n|} < \infty$$

   admits a generating function, defined for $|z| < R = 1/M$.
3. Every such sequence is uniquely determined by its generating function.

We can thus use all the standard operations on sequences and their generating functions, without caring about definition, convergence, etc., provided we do so under the tacit assumption that we are in a "small enough" circle centered in the origin of the complex plane.

Basic operations with generating functions

Let $\langle g_0, g_1, g_2, \ldots \rangle$ and $\langle h_0, h_1, h_2, \ldots \rangle$ be sequences of complex numbers.
Let $G(z) = \sum_{n \geqslant 0} g_n z^n$ and $H(z) = \sum_{n \geqslant 0} h_n z^n$ be their generating functions.
The following operations are well defined in a neighborhood of the origin:

sequence	generic term	g.f.
$\langle \alpha g_0 + \beta h_0, \alpha g_1 + \beta h_1, \alpha g_2 + \beta h_2, \ldots \rangle$	$\alpha g_n + \beta h_n$	$\alpha G(z) + \beta H(z)$
$\langle 0, \ldots, 0, g_0, g_1, \ldots \rangle$	$g_{n-m}[n \geqslant m]$	$z^m G(z)$
$\langle g_m, g_{m+1}, g_{m+2}, \ldots \rangle$	g_{n+m}	$\dfrac{G(z) - \sum_{k=0}^{m-1} g_k z^k}{z^m}$
$\langle a_1, 2a_2, 3a_3, \ldots \rangle$	$(n+1) g_{n+1}$	$G'(z)$
$\langle 0, a_0, \frac{a_1}{2}, \ldots \rangle$	$\frac{g_{n-1}}{n}[n > 0]$	$\int_0^z G(w)dw$
$\langle g_0 h_0, g_0 h_1 + g_1 h_0, g_0 h_2 + g_1 h_1 + g_2 h_0, \ldots \rangle$	$\sum_{k=0}^n g_k h_{n-k}$	$G(z) \cdot H(z)$

where:

- undefined $\cdot\, 0 = 0$ by convention.
- $\int_0^z G(w)dw = \int_0^1 zG(tz)dt = \Gamma(z)$ with $\Gamma' = G$ and $\Gamma(0) = 0$.

Derivatives of the generating function

Theorem

If $G(z) = \sum_{n \geqslant 0} g_n z^n$, then for every $k \geqslant 0$,

$$G^{(k)}(z) = \sum_{n \geqslant 0} (n+k)^{\underline{k}} g_{n+k} z^n$$

Derivatives of the generating function

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The thesis is true for $k = 0$ as $n^{\underline{0}}$ is an empty product.
If the thesis is true for k, then

$$\begin{aligned} G^{(k+1)}(z) &= \sum_{n \geqslant 0} (n+k)^{\underline{k}} n g_{n+k} z^{n-1} [n \geqslant 1] \\ &= \sum_{n \geqslant 0} (n+1+k)^{\underline{k}} (n+1)^{\underline{1}} g_{n+1+k} z^n \\ &= \sum_{n \geqslant 0} (n+1+k)^{\underline{k+1}} g_{n+1+k} z^n \end{aligned}$$

Derivatives of the generating function

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Corollary

For every $n \geqslant 0$,

$$g_n = \frac{n^{\underline{n}}}{n!} g_n = \frac{1}{n!} \sum_{k \geqslant 0} (n+k)^{\underline{n}} g_{n+k} 0^k = \frac{G^{(n)}(0)}{n!}$$

Next subsection

Counting with Generating Functions

Choosing k-subset from n-set

Binomial theorem yields:

$$\left\langle \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots \right\rangle \longleftrightarrow$$
$$\longleftrightarrow \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$$

- Thus, the coefficient of x^k in $(1+x)^n$ is the number of ways to choose k distinct items from a set of size n.
- For example, the coefficient of x^2 is , the number of ways to choose 2 items from a set with n elements.
- Similarly, the coefficient of x^{n+1} is the number of ways to choose $n+1$ items from a n-set, which is zero.

Counting with Generating Functions

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Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is the function $A(x)$ that's expansion into power series has coefficient $a_i = 1$ of x^i iff i can be selected into the subset, otherwise $a_i = 0$.

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

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Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If any even number of elements can be selected:

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

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- If any even number of elements can be selected:

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

- If any positive even number of elements can be selected:

$$A(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2}$$

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- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If any number of elements multiple of 5 can be selected:

$$A(x) = 1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5}$$

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Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

- If at most four elements can be selected:

$$A(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

- If at most one element can be selected:

$$A(x) = \frac{1-x^2}{1-x} = 1 + x$$

Counting elements of two sets

Convolution Rule

Let $A(x)$ be the generating function for selecting an item from (multi)set $\mathscr{A}$, and let $B(x)$, be the generating function for selecting an item from (multi)set $\mathscr{B}$. If $\mathscr{A}$ and $\mathscr{B}$ are disjoint, then the generating function for selecting items from the union $\mathscr{A} \cup \mathscr{B}$ is the product $A(x) \cdot B(x)$.

Proof. To count the number of ways to select n items from $\mathscr{A} \cup \mathscr{B}$ we have to select j items from $\mathscr{A}$ and $n-j$ items from $\mathscr{B}$, where $j \in \{0, 1, 2, \ldots, n\}$.
Summing over all the possible values of j gives a total of

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

ways to select n items from $\mathscr{A} \cup \mathscr{B}$. This is precisely the coefficient of x^n in the series for $A(x) \cdot B(x)$ Q.E.D.

Warmup: The old lady and her pets

The problem

When a certain old lady walks her pets, she brings with her:

- three, four, or five dogs;
- a cage with several pairs of rabbits;
- and (sometimes) her crocodile.

In how many ways can she walk n pets, for $n \geqslant 0$?

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Using generating functions

Let $D(z)$, $R(z)$, and $C(z)$ be the generating functions of the number of ways the old lady can walk dogs, rabbits, and crocodiles, respectively:

$$D(z) = z^3 + z^4 + z^5\,; \;\; R(z) = 1 + z^2 + z^4 + \cdots = \frac{1}{1 - z^2}\,; \;\; C(z) = 1 + z$$

The generating function $A(z)$ of the number of ways the old lady can walk pets is thus:

$$A(z) = D(z) \cdot R(z) \cdot C(z) = \frac{z^3 + z^4 + z^5}{1 - z}$$

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Solution

For $m \geqslant 0$ integer, $G(z) = z^m$ is the generating function of $g_n = [n = m]$.
$G(z) = 1/(1-z)$ is the generating function of $g_n = 1$.
Then for every $n \geqslant 0$, the number of ways the old lady can walk her pets is:

$$a_n = [z^n]A(z) = \sum_{m=3}^{5} \sum_{k=0}^{n} [k = m] = \sum_{m=3}^{5} [n \geqslant m]$$

For example, for $n = 6$ the old lady has three choices:

- three pairs of rabbits;
- four dogs and one pair of rabbits;
- three dogs, one pair of rabbits, and the crocodile.

Next subsection

Generating function for arbitrary binomial coefficients

Theorem (Generalized binomial theorem)

For every $r \in \mathbb{R}$,

$$(1+z)^r = \sum_{n \geqslant 0} \binom{r}{n} z^n$$

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For every $r \in \mathbb{R}$,

$$(1+z)^r = \sum_{n \geqslant 0} \binom{r}{n} z^n$$

Indeed, let $G(z) = (1+z)^r$ where r is an arbitrary real number:

- By differentiating $n \geqslant 0$ times, $G^{(n)}(z) = r \cdots (r-1) \cdots (r-n+1) \cdot (1+z)^{r-n}$.
- Then,

$$\frac{G^{(n)}(0)}{n!} = \frac{r^{\underline{n}}}{n!} = \binom{r}{n}$$

- As n is arbitrary and the correspondence between sequences and generating functions is one-to-one, the thesis follows.

Generating function for arbitrary binomial coefficients

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$$\frac{G^{(n)}(0)}{n!} = \frac{r^{\underline{n}}}{n!} = \binom{r}{n}$$

- As n is arbitrary and the correspondence between sequences and generating functions is one-to-one, the thesis follows.

Example

$$\sqrt{1+z} = \sum_{n \geqslant 0} \binom{1/2}{n} z^n$$

Vandermonde's identity

Binomial theorem provides the generating function:

$$(1+z)^r = \sum_{n \geqslant 0} \binom{r}{n} z^n \qquad \text{and} \qquad (1+z)^s = \sum_{n \geqslant 0} \binom{s}{n} z^n$$

Convolution yields:

$$\begin{aligned}
\sum_{n \geqslant 0} \binom{r+s}{n} z^n &= (1+z)^{r+s} \\
&= (1+z)^r \cdot (1+z)^s \\
&= \left(\sum_{n \geqslant 0} \binom{r}{n} z^n \right) \cdot \left(\sum_{n \geqslant 0} \binom{s}{n} z^n \right) \\
&= \sum_{n \geqslant 0} \left(\sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} \right) z^n
\end{aligned}$$

Vandermonde's identity (2)

Equating coefficients of z^k on both sides of this equation gives:

Vandermonde's convolution:

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

Convolution yields

$$\binom{9}{2} = \binom{3}{0}\binom{6}{2} + \binom{3}{1}\binom{6}{1} + \binom{3}{2}\binom{6}{0}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

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1

1 1

1 2 1

$\boxed{1}$ $\boxed{3}$ $\boxed{3}$ 1

1 4 6 4 1

1 5 10 10 5 1

$\boxed{1}$ $\boxed{6}$ $\boxed{15}$ 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 $\boxed{36}$ 84 126 126 84 36 9 1

$$\binom{9}{2} = \binom{3}{0}\binom{6}{2} + \binom{3}{1}\binom{6}{1} + \binom{3}{2}\binom{6}{0}$$

Vandermonde's identity (3)

Special case $r = s = n$ yields:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k \geqslant 0} \binom{n}{k}^2$$

Example: $\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 = \binom{8}{4}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 $\boxed{70}$ 56 28 8 1

1
1 1
1 4 1
1 9 9 1
$\boxed{1}$ $\boxed{16}$ $\boxed{36}$ $\boxed{16}$ $\boxed{1}$

$1 + 16 + 36 + 16 + 1 = 70$

Vandermonde's identity (3)

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$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&\boxed{70}&&56&&28&&8&&1
\end{array}$$

$$\begin{array}{ccccccccc}
&&&&1&&&& \\
&&&1&&1&&& \\
&&1&&4&&1&& \\
&1&&9&&9&&1& \\
\boxed{1}&&\boxed{16}&&\boxed{36}&&\boxed{16}&&\boxed{1}
\end{array}$$

$1 + 16 + 36 + 16 + 1 = 70$

Sequence $\left\langle \binom{m}{0}, 0, -\binom{m}{1}, 0, \binom{m}{2}, 0, -\binom{m}{3}, 0, \binom{m}{4}, 0, \dots, \right\rangle$

Let's take sequences

$$\left\langle \binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \dots, \binom{m}{n}, \dots \right\rangle \longleftrightarrow F(z) = (1+z)^m$$

and

$$\left\langle \binom{m}{0}, -\binom{m}{1}, \binom{m}{2}, \dots, (-1)^n \binom{m}{n}, \dots \right\rangle \longleftrightarrow G(z) = F(-z) = (1-z)^m$$

Then the convolution corresponds to the function $(1+z)^m(1-z)^m = (1-z^2)^m$ that gives the identity of binomial coefficients:

$$\sum_{j=0}^{n} \binom{m}{j} \binom{m}{n-j} (-1)^j = (-1)^{n/2} \binom{m}{n/2} [n \text{ is even}]$$

Other useful binomial identities

Sign change and falling powers

$$(-1)^n r^{\underline{n}} = (n-r-1)^{\underline{n}} \;\; \forall r \in \mathbb{R} \; \forall n \geqslant 0$$

Proof: $(-1)^n \cdot r \cdot (r-1) \cdots (r-n+2) \cdot (r-n+1) = (n-r-1) \cdot (n-r-2) \cdots (1-r) \cdot (-r)$

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Generating function for binomial coefficients with upper index increasing

For every $r \geqslant 0$,

$$\frac{1}{(1-z)^{r+1}} = \sum_{n \geqslant 0} (-1)^n \binom{-1-r}{n} z^n = \sum_{n \geqslant 0} \binom{r+n}{n} z^n$$

In addition, if $r = m$ is an integer,

$$\frac{1}{(1-z)^{m+1}} = \sum_{n \geqslant 0} \binom{m+n}{n} z^n = \sum_{n \geqslant 0} \binom{m+n}{m} z^n$$

and by shifting,

$$\frac{z^m}{(1-z)^{m+1}} = \sum_{n \geqslant 0} \binom{m+n}{m} z^{m+n} = \sum_{n \geqslant 0} \binom{n}{m} z^n$$

Generating functions cheat sheet

$$
\begin{aligned}
e^z &= \sum_{n\geqslant 0} \frac{z^n}{n!} \\
\frac{1}{1-\alpha z} &= \sum_{n\geqslant 0} \alpha^n z^n \\
(1+z)^r &= \sum_{n\geqslant 0} \binom{r}{n} z^n, \quad r \in \mathbb{R} \\
\frac{1}{(1-z)^{r+1}} &= \sum_{n\geqslant 0} \binom{r+n}{n} z^n, \quad r \in \mathbb{R} \\
\frac{1}{(1-z)^{m+1}} &= \sum_{n\geqslant 0} \binom{m+n}{m} z^n, \quad m \in \mathbb{N} \\
\frac{z^m}{(1-z)^{m+1}} &= \sum_{n\geqslant 0} \binom{n}{m} z^n, \quad m \in \mathbb{N} \\
\ln \frac{1}{1-z} &= \sum_{n\geqslant 1} \frac{z^n}{n} \\
\frac{1}{1-z} \ln \frac{1}{1-z} &= \sum_{n\geqslant 1} H_n z^n
\end{aligned}
$$

Generating functions and recurrences

Problem

Solve the recurrence:

$$\begin{aligned} U_0 &= 1\,, \\ U_n &= U_{n-1} + n + 3 \;\; \forall n \geqslant 1\,. \end{aligned}$$

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Setting up the problem

Let $G(z) = \sum_{n \geqslant 0} U_n z^n$. The initial condition tells us that $G(0) = 1$. For $n \geqslant 1$:

- U_{n-1} is the coefficient of z^n in $zG(z)$.
- n is the coefficient of z^n in $\frac{z}{(1-z)^2}$, because $n+1$ is of $\frac{1}{(1-z)^2}$.
- 1 is the coefficient of z^n in $\frac{1}{1-z} - 1 = \dfrac{z}{1-z}$.

  (We must remove the constant term, because the recurrence only holds for $n \geqslant 1$.)

Then $G(z)$ satisfies the following functional equation:

$$G(z) = 1 + zG(z) + \frac{z}{(1-z)^2} + \frac{3z}{1-z}$$

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Solve the recurrence:

$$\begin{aligned} U_0 &= 1\,, \\ U_n &= U_{n-1} + n + 3 \;\; \forall n \geqslant 1\,. \end{aligned}$$

Expressing as power series

We can rewrite the functional equation as:

$$G(z) = \frac{1}{1-z} + \frac{z}{(1-z)^3} + \frac{3z}{(1-z)^2}$$

We know that

$$\frac{1}{1-z} = \sum_{n \geqslant 0} z^n \text{ and } \frac{z}{(1-z)^2} = \sum_{n \geqslant 0} n z^n$$

For the middle summand, we observe:

$$\frac{z}{(1-z)^3} = \frac{z}{2} \frac{d}{dz} \frac{1}{(1-z)^2} = \frac{z}{2} \sum_{n \geqslant 1} (n+1) n z^{n-1} = \sum_{n \geqslant 0} \binom{n+1}{2} z^n$$

Generating functions and recurrences

Problem

Solve the recurrence:

$$\begin{array}{rcl} U_0 & = & 1\,, \\ U_n & = & U_{n-1} + n + 3 \;\; \forall n \geqslant 1\,. \end{array}$$

Reconstructing the coefficients

Our functional equation is thus equivalent to:

$$\sum_{n \geqslant 0} U_n z^n = \sum_{n \geqslant 0} z^n + \sum_{n \geqslant 0} \binom{n+1}{2} z^n + 3 \sum_{n \geqslant 0} \sum_{n \geqslant 0} n z^n$$

By linearity and uniqueness it must then be:

$$U_n = 1 + \frac{n(n+1)}{2} + 3n \;\; \forall n \geqslant 0$$