Integer Fuctions

ITT9131 Konkreetne Matemaatika

Chapter Three

- Floors and Ceilings
- Floor/Ceiling Applications
- Floor/Ceiling Recurrences
- ‘mod’: The Binary Operation
- Floor/Ceiling Sums

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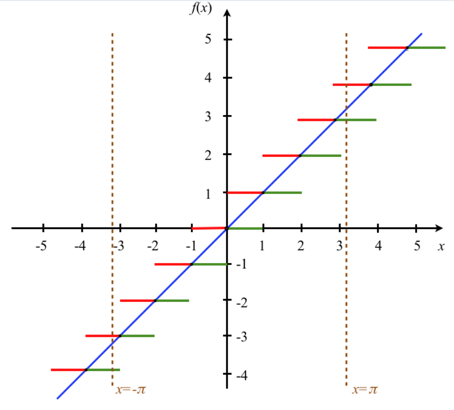
Floors and Ceilings

Definition

- The floor $\lfloor x \rfloor$ is the greatest integer less than or equal to x ;
- The ceiling $\lceil x \rceil$ is the least integer greater than or equal to x .



$$\lfloor \pi \rfloor = 3 \qquad \lfloor -\pi \rfloor = -4$$
$$\lceil \pi \rceil = 4 \qquad \lceil -\pi \rceil = -3$$

Floors and Ceilings

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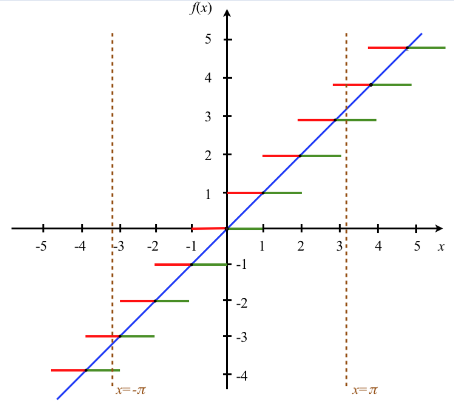

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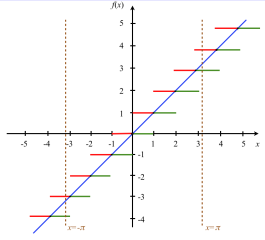
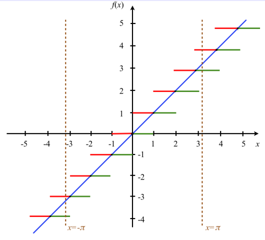
Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$



Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$



① $\lfloor x \rfloor = x = \lceil x \rceil$ iff $x \in \mathbb{Z}$

② $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$

③ $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$

④ $\lceil x \rceil - \lfloor x \rfloor = [x \notin \mathbb{Z}]$

Warmup: the generalized Dirichlet box principle

Statement of the principle

Let m and n be positive integers. If n items are stored into m boxes, then:

- At least one box will contain at least $\lceil n/m \rceil$ objects.
- At least one box will contain at most $\lfloor n/m \rfloor$ objects.

Warmup: the generalized Dirichlet box principle

Statement of the principle

Let m and n be positive integers. If n items are stored into m boxes, then:

- At least one box will contain at least $\lceil n/m \rceil$ objects.
- At least one box will contain at most $\lfloor n/m \rfloor$ objects.

Proof

By contradiction, assume each of the m boxes contains fewer than $\lceil n/m \rceil$ objects. Then

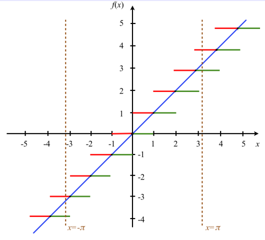
$$n \leqslant m \cdot \left(\left\lceil \frac{n}{m} \right\rceil - 1 \right) \text{ or equivalently}, \ \frac{n}{m} + 1 \leqslant \left\lceil \frac{n}{m} \right\rceil :$$

which is impossible.
Similarly, if each of the m boxes contained more than $\lfloor n/m \rfloor$ objects, we would have

$$n \geqslant m \cdot \left(\left\lfloor \frac{n}{m} \right\rfloor + 1 \right) \text{ or equivalently}, \ \frac{n}{m} - 1 \geqslant \left\lfloor \frac{n}{m} \right\rfloor :$$

which is also impossible.

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$ (cont.)



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In the following properties $x \in \mathbb{R}$ and $n \in \mathbb{Z}$:

⑤ $\lfloor x \rfloor = n$ iff $n \leqslant x < n + 1$

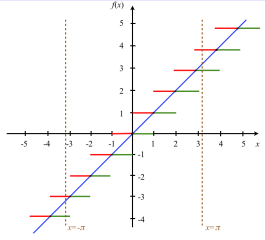
⑥ $\lfloor x \rfloor = n$ iff $x - 1 < n \leqslant x$

⑦ $\lceil x \rceil = n$ iff $n - 1 < x \leqslant n$

⑧ $\lceil x \rceil = n$ iff $x \leqslant n < x + 1$

⑨ $\lfloor x + n \rfloor = \lfloor x \rfloor + n$, but $\lfloor nx \rfloor \neq n \lfloor x \rfloor$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$ (cont.)



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More properties:

⑩ $x < n$ iff $\lfloor x \rfloor < n$

⑪ $n < x$ iff $n < \lceil x \rceil$

⑫ $x \leqslant n$ iff $\lceil x \rceil \leqslant n$

⑬ $n \leqslant x$ iff $n \leqslant \lfloor x \rfloor$

Generalization of the property #9

Theorem

$$\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } 0 \leqslant \{x\}+\{y\} < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } 1 \leqslant \{x\}+\{y\} < 2 \end{cases}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Proof. Let $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$

$$\begin{aligned} \lfloor x+y \rfloor &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor \end{aligned}$$

and

$$\lfloor \{x\}+\{y\} \rfloor = \begin{cases} 0, & \text{if } 0 \leqslant \{x\}+\{y\} < 1 \\ 1, & \text{if } 1 \leqslant \{x\}+\{y\} < 2 \end{cases}$$

Q.E.D.

Warmup: When is $\lfloor nx \rfloor = n \lfloor x \rfloor$?

The problem

Give a necessary and sufficient condition on n and x so that

$$\lfloor nx \rfloor = n \lfloor x \rfloor$$

where n is a positive integer.

Warmup: When is $\lfloor nx \rfloor = n \lfloor x \rfloor$?

The problem

Give a necessary and sufficient condition on n and x so that

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where n is a positive integer.

The solution

Write $x = \lfloor x \rfloor + \{x\}$. Then

$$\lfloor nx \rfloor = \lfloor n \lfloor x \rfloor + n\{x\} \rfloor = n \lfloor x \rfloor + \lfloor n\{x\} \rfloor$$

As $\{x\}$ is nonnegative, so is $\lfloor n\{x\} \rfloor$. Then

$$\lfloor nx \rfloor = n \lfloor x \rfloor \text{ if and only if } \{x\} < 1/n$$

Next section

Floor/Ceiling Applications

Theorem

The binary representation of a natural number $n > 0$ has $m = \lfloor \log_2 n \rfloor + 1$ bits.

Proof.

$$n = \underbrace{a_{m-1}2^{m-1} + a_{m-2}2^{m-2} + \cdots + a_1 2 + a_0}_{m \text{ bits}}, \text{ where } a_{m-1} = 1$$

Thus, $2^{m-1} \leqslant n < 2^m$, that gives $m - 1 \leqslant \log_2 n < m$. The last formula is valid if and only if $\lfloor \log_2 n \rfloor = m - 1$. Q.E.D.

Example: $n = 35 = 100011_2$

$$m = \lfloor \log_2 35 \rfloor + 1 = \lfloor \log_2 32 \rfloor + 1 = 5 + 1 = 6$$

Floor/Ceiling Applications

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Floor/Ceiling Applications (2)

Theorem

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing function with the property that $f(x) \in \mathbb{Z}$ implies that $x \in \mathbb{Z}$. Then

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \qquad \text{and} \qquad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

whenever $f(x)$, $f(\lfloor x \rfloor)$, and $f(\lceil x \rceil)$ are all defined.

Proof. (for the ceiling function)

- The case $x = \lceil x \rceil$ is trivial.
- Otherwise $x < \lceil x \rceil$, and $f(x) < f(\lceil x \rceil)$ since f is increasing. Hence, $\lceil f(x) \rceil \leqslant \lceil f(\lceil x \rceil) \rceil$ since $\lceil : \rceil$ is non-decreasing.
- If $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$, as f is continuous, by the intermediate value theorem there exists a number y such that $y \in [x, \lceil x \rceil)$ and $f(y) = \lceil f(x) \rceil$: such y is an integer, because of f's special property, so actually $x < y < \lceil x \rceil$.
- But there cannot be an integer strictly between x and $\lceil x \rceil$. This contradiction implies that we must have $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$.

Q.E.D.

Floor/Ceiling Applications (2a)

Example

- $\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$
- $\left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$
- $\lceil \lceil \lceil x/10 \rceil /10 \rceil /10 \rceil = \lceil x/1000 \rceil$
- $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$

Floor/Ceiling Applications (2a)

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- $\lceil \lceil \lceil x/10 \rceil /10 \rceil /10 \rceil = \lceil x/1000 \rceil$
- $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$

In contrast:

$$\left\lceil \sqrt{\lfloor x \rfloor} \right\rceil \neq \lceil \sqrt{x} \rceil$$

Floor/Ceiling Applications (3) : Intervals

For Real numbers $\alpha \neq \beta$

Interval	Integers contained	Restrictions
$[\alpha..\beta]$	$\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$	$\alpha \leqslant \beta$
$[\alpha..\beta)$	$\lceil\beta\rceil - \lceil\alpha\rceil$	$\alpha \leqslant \beta$
$(\alpha..\beta]$	$\lfloor\beta\rfloor - \lfloor\alpha\rfloor$	$\alpha \leqslant \beta$
$(\alpha..\beta)$	$\lceil\beta\rceil - \lfloor\alpha\rfloor - 1$	$\alpha < \beta$

Floor/Ceiling Applications (3) : Spectra

Definition

The spectrum of a real number α is an infinite multiset of integers

$$\mathrm{Spec}(\alpha) = \{\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \ldots\} = \{\lfloor n\alpha\rfloor \mid n \geq 1\}$$

Theorem

If $\alpha \neq \beta$ then $\mathrm{Spec}(\alpha) \neq \mathrm{Spec}(\beta)$.

Proof. For, assuming without loss of generality that $\alpha < \beta$, there's a positive integer m such that $m(\beta - \alpha) \geqslant 1$. Hence $m\beta - m\alpha \geqslant 1$, and $\lfloor m\beta\rfloor > \lfloor m\alpha\rfloor$. Thus $\mathrm{Spec}(\beta)$ has fewer than m elements which are $\leqslant \lfloor m\alpha\rfloor$, while $\mathrm{Spec}(\alpha)$ has at least m such elements. Q.E.D.

Example.

$$\mathrm{Spec}(\sqrt{2}) = \{1,2,4,5,7,8,9,11,12,14,15,16,18,19,21,22,24,\ldots\}$$
$$\mathrm{Spec}(2+\sqrt{2}) = \{3,6,10,13,17,20,23,27,30,34,37,40,44,47,51,\ldots\}$$

Floor/Ceiling Applications (3a) : Spectra

The number of elements in $\mathrm{Spec}(\alpha)$ that are $\leqslant n$:

$$\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor k\alpha \rfloor \leqslant n] \\
&= \sum_{k>0} [\lfloor k\alpha \rfloor < n+1] \\
&= \sum_{k>0} [k\alpha < n+1] \\
&= \sum_{k} [0 < k < (n+1)/\alpha] \\
&= \lceil (n+1)/\alpha \rceil - 1
\end{aligned}$$

Floor/Ceiling Applications (3b) : Spectra

Let's compute (for any $n > 0$):

$$\begin{aligned}
N(\sqrt{2},n)+N(2+\sqrt{2},n) &= \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1 \\
&= \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor \\
&= \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \\
&= (n+1)\underbrace{\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} \right)}_{=1} - \underbrace{\left(\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \right)}_{=1} \\
&= n+1-1 = n
\end{aligned}$$

Corollary

The spectra $\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$ form a partition of the positive integers.

Floor/Ceiling Applications (3b) : Spectra

Let's compute (for any $n > 0$):

$$\begin{aligned}
N(\sqrt{2}, n) + N(2+\sqrt{2}, n) &= \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1 \\
&= \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor \\
&= \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \\
&= (n+1) \underbrace{\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} \right)}_{=1} - \underbrace{\left(\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \right)}_{=1} \\
&= n + 1 - 1 = n
\end{aligned}$$

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The spectra $\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$ form a partition of the positive integers.

Next section

Floor/Ceiling Recurrences: Examples

The Knuth numbers:

$$\begin{aligned} K_0 &= 1; \\ K_{n+1} &= 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}) \qquad \text{for } n \geqslant 0. \end{aligned}$$

The sequence begins as

$$K = \langle 1, 3, 3, 4, 7, 7, 7, 9, 9, 10, 13, \dots \rangle$$

Floor/Ceiling Recurrences: Examples

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The sequence begins as

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Merge sort $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$ records, number of comparisons:

$$
\begin{aligned}
f_1 &= 0; \\
f_{n+1} &= f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + n - 1 \qquad \text{for } n > 1.
\end{aligned}
$$

The sequence begins as

$$f = \langle 0, 1, 3, 5, 8, 11, 14, 17, 21, 25, 29, 33 \ldots \rangle$$

Floor/Ceiling Recurrences: More Examples

The Josephus problem numbers:

$$J(1) = 1;$$
$$J(n) = 2J(\lfloor n/2 \rfloor) - (-1)^n \qquad \text{for } n > 1.$$

The sequence begins as

$$J = \langle 1, 1, 3, 1, 3, 5, 7, 1, 3, 5, \ldots \rangle$$

Generalization of Josephus problem

Josephus problem in general: from n elements, every q-th is circularly eliminated. The element with number $J_q(n)$ will survive.

Theorem

$$J_q(n) = qn + 1 - D_k$$

where k is as small as possible such that $D_k > (q-1)n$ and D_k is computed using the following recurrent relation:

$$D_0 = 1;$$

$$D_n = \left\lceil \frac{q}{q-1} D_{n-1} \right\rceil \qquad \textit{for } n > 0.$$

For example, if $q = 5$ and $n = 12$

$$D = \langle 1, 2, 3, 4, 5, 7, 9, 12, 15, 19, 24, 30, 38, 48, 60, 75 \ldots \rangle$$

Then $(q-1)n = 4 \cdot 12 = 48$, the proper D_k is $D_{14} = 60$, and

$$J_5(12) = 5 \cdot 12 + 1 - D_{14} = 60 + 1 - 60 = 1$$

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Proof of the Theorem

Whenever a person is passed over, we can assign a new number, as in the example below fo $n = 12, q = 5$

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16		17	18	19	20		21	22
23	24		25		26	27	28			29	30
31	32				33	34	35			36	
37	38				39	40				41	
42	43				44					45	
46	47				48						
49	50				51						
52					53						
54					55						
56											
57											
58											
59											
60											

Denoting by N and N' succeeding elements in a column, we get

$$N = \left\lfloor \frac{N' - n - 1}{q - 1} \right\rfloor + N' - n$$

Proof of the Theorem (2)

Denoting by $D = qn + 1 - N$ and $D' = qn + 1 - N'$, we obtain for the formula

$$N = \left\lfloor \frac{N' - n - 1}{q - 1} \right\rfloor + N' - n$$

another form:

$$qn + 1 - D = \left\lfloor \frac{qn + 1 - D' - n - 1}{q - 1} \right\rfloor + qn + 1 - D' - n$$

Let us transform this:

$$\begin{aligned}
D &= qn + 1 - \left\lfloor \frac{qn + 1 - D' - n - 1}{q - 1} \right\rfloor - qn - 1 + D' + n \\
&= D' + n - \left\lfloor \frac{n(q-1) - D'}{q - 1} \right\rfloor \\
&= D' + n - \left\lfloor n - \frac{D'}{q - 1} \right\rfloor \\
&= D' - \left\lfloor \frac{-D'}{q - 1} \right\rfloor \\
&= D' + \left\lceil \frac{D'}{q - 1} \right\rceil \\
&= \left\lceil \frac{q}{q - 1} D' \right\rceil
\end{aligned}$$

Next section

'mod': The Binary Operation

If n and m are positive integers

Write $n = q \cdot m + r$ with $q, r \in \mathbb{N}$ and $0 \leqslant r < m$. Then:

$$q = \lfloor n/m \rfloor \text{ and } r = n - m \cdot \lfloor n/m \rfloor = n \bmod m$$

If x and y are real numbers

We follow the same idea and set:

$$x \bmod y = x - y \cdot \lfloor x/y \rfloor \ \forall x, y \in \mathbb{R},\ y \neq 0$$

Note that, with this definition:

$$\begin{array}{lclclcl}
5 \bmod 3 & = & 5 - 3 \cdot \lfloor 5/3 \rfloor & = & 5 - 3 \cdot 1 & = & 2 \\
5 \bmod -3 & = & 5 - (-3) \cdot \lfloor 5/(-3) \rfloor & = & 5 + 3 \cdot (-2) & = & -1 \\
-5 \bmod 3 & = & -5 - 3 \cdot \lfloor -5/3 \rfloor & = & -5 - 3 \cdot (-2) & = & 1 \\
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\end{array}$$

For $y = 0$ we want to respect the general rule that $x - (x \bmod y) \in y\mathbb{Z} = \{yk \mid k \in \mathbb{Z}\}$. This is done by:

$$x \bmod 0 = x$$

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‘mod’: The Binary Operation

If n and m are positive integers

Write $n = q \cdot m + r$ with $q, r \in \mathbb{N}$ and $0 \leqslant r < m$. Then:

$$q = \lfloor n/m \rfloor \text{ and } r = n - m \cdot \lfloor n/m \rfloor = n \bmod m$$

If x and y are real numbers

We follow the same idea and set:

$$x \bmod y = x - y \cdot \lfloor x/y \rfloor \ \forall x, y \in \mathbb{R},\ y \neq 0$$

Note that, with this definition:

$$\begin{array}{lclclcl}
5 \bmod 3 & = & 5 - 3 \cdot \lfloor 5/3 \rfloor & = & 5 - 3 \cdot 1 & = & 2 \\
5 \bmod -3 & = & 5 - (-3) \cdot \lfloor 5/(-3) \rfloor & = & 5 + 3 \cdot (-2) & = & -1 \\
-5 \bmod 3 & = & -5 - 3 \cdot \lfloor -5/3 \rfloor & = & -5 - 3 \cdot (-2) & = & 1 \\
-5 \bmod -3 & = & -5 - (-3) \cdot \lfloor -5/(-3) \rfloor & = & -5 + 3 \cdot 1 & = & -2
\end{array}$$

For $y = 0$ we want to respect the general rule that $x - (x \bmod y) \in y\mathbb{Z} = \{yk \mid k \in \mathbb{Z}\}$. This is done by:

$$x \bmod 0 = x$$

Properties of the mod operation

$x = \lfloor x \rfloor + x \bmod 1$

For $y = 1$ it is $x \bmod 1 = x - 1 \cdot \lfloor x/1 \rfloor = x - \lfloor x \rfloor$.

The distributive law: $c(x \bmod y) = cx \bmod cy$

If $c = 0$ both sides vanish; if $y = 0$ both sides equal cx. Otherwise:

$$c(x \bmod y) = c(x - y \lfloor x/y \rfloor) = cx - cy \lfloor cx/cy \rfloor = cx \bmod cy$$

Warmup: Solve the following recurrence

$$
\begin{aligned}
X_n &= n && \text{for } 0 \leqslant n < m\,, \\
X_n &= X_{n-m} + 1 && \text{for } n \geqslant m\,.
\end{aligned}
$$

Warmup: Solve the following recurrence

$$X_n = n \qquad \text{for } 0 \leqslant n < m\,,$$
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Solution

We plot the first values when $m = 4$:

n	0	1	2	3	4	5	6	7	8	9
X_n	0	1	2	3	1	2	3	4	2	3

We conjecture that:

$$\text{if } n = qm + r \text{ with } q, r \in \mathbb{N} \text{ and } 0 \leqslant r < m \text{ then } X_n = q + r\ :$$

which clearly yields $X_n = \lfloor n/m \rfloor + n \bmod m$.

- Induction base: True for $n = 0, 1, \ldots, m-1$.
- Inductive step: Let $n \geq m$. If $X_{n'} = q' + r'$ for every $n' = q'm + r' < n = qm + r$, then:

$$X_n = X_{n-m} + 1 = X_{(q-1)m+r} + 1 = q - 1 + r + 1 = q + r$$

Next section

Floor/Ceiling Sums

Example: Find the sum $\sum_{0\leqslant k<n}\lfloor\sqrt{k}\rfloor$ in its closed form:

$$\begin{aligned}
\sum_{0\leqslant k<n}\lfloor\sqrt{k}\rfloor &= \sum_{k,m\geqslant 0} m[k<n][m=\lfloor\sqrt{k}\rfloor] \\
&= \sum_{k,m\geqslant 0} m[k<n][m\leqslant\sqrt{k}<m+1] \\
&= \sum_{k,m\geqslant 0} m[k<n][m^2\leqslant k<(m+1)^2] \\
&= \underbrace{\sum_{k,m\geqslant 0} m[m^2\leqslant k<(m+1)^2\leqslant n]+}_{=S_1} \\
&\qquad + \underbrace{\sum_{k,m\geqslant 0} m[m^2\leqslant k<n<(m+1)^2]}_{=S_2}
\end{aligned}$$

Floor/Ceiling Sums (2)

Example continues …

Case $n = a^2$, for a value $a \in \mathbb{N}$

- $S_2 = 0$
- $$\begin{aligned} S_1 &= \sum_{k,m\geqslant 0} m[m^2 \leqslant k < (m+1)^2 \leqslant a^2] \\ &= \sum_{m\geqslant 0} m((m+1)^2 - m^2)[m+1 \leqslant a] \\ &= \sum_{m\geqslant 0} m(2m+1)[m < a] \\ &= \sum_{m\geqslant 0} (2m(m-1)+3m)[m < a] \\ &= \sum_{m\geqslant 0} (2m^{\underline{2}} + 3m^{\underline{1}})[m < a] = \sum_0^a (2m^{\underline{2}} + 3m^{\underline{1}})\delta m \\ &= (\frac{2}{3}m^{\underline{3}} + \frac{3}{2}m^{\underline{2}})\Big|_0^a = \frac{2}{3}a(a-1)(a-2) + \frac{3}{2}a(a-1) \\ &= \frac{2}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a \end{aligned}$$

Floor/Ceiling Sums (3)

Example continues …

Case $n \neq b^2$, for any integer b; let $a = \lfloor \sqrt{n} \rfloor$

- For $0 \leqslant k < a^2$ we get $S_1 = \frac{2}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a$ and $S_2 = 0$, as before;
- For $a^2 \leqslant k < n$, it is valid that $S_1 = 0$ and

$$\begin{aligned} S_2 &= \sum_{k,m \geqslant 0} m[m^2 \leqslant k < n < (m+1)^2] \\ &= \sum_k a[a^2 \leqslant k < n] \\ &= a \sum_k [a^2 \leqslant k < n] \\ &= a(n - a^2) = an - a^3 \end{aligned}$$

To summarize:

$$\begin{aligned} \sum_{0 \leqslant k < n} \lfloor \sqrt{k} \rfloor &= \frac{2}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a + an - a^3 \\ &= an - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a, \qquad \text{where } a = \lfloor \sqrt{n} \rfloor \end{aligned}$$

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Case $n \neq b^2$, for any integer b; let $a = \lfloor \sqrt{n} \rfloor$

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To summarize:

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