ITT9131 Konkreetne Matemaatika

Concrete mathematics

Конкретная математика

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TTÜ arvutiteaduse instituut, teoreetilise informaatika õppetool

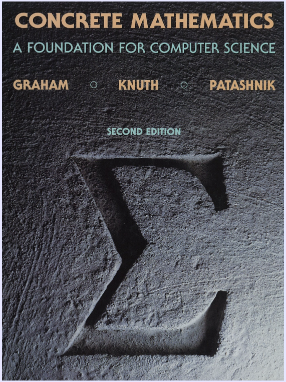
Küberneetika Instituudi tarkvarateaduse laboratoorium

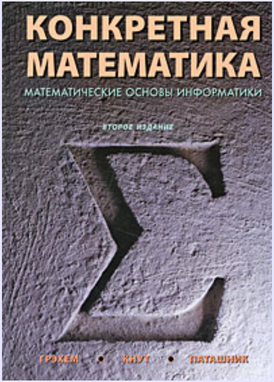
2016./17. õppeaasta sügissemester

CONtinuous + disCRETE

MATHEMATICS

The book





http://www-cs-faculty.stanford.edu/~uno/gkp.html

Concrete Mathematics is ...

- the controlled manipulation of mathematical formulas
- using a collection of techniques for solving problems

Goals of the book:

- to introduce the mathematics that supports advanced computer programming and the analysis of algorithms
- to provide a solid and relevant base of mathematical skills - the skills needed
  - to solve complex problems
  - to evaluate horrendous sums
  - to discover subtle patterns in data

Our additional goals

- to get acquainted with well-known and popular literature in CS and Math
- to develop mathematical skills, formulating complex problems mathematically
- to practice presentation of results (solutions of mathematical problems)

Contents of the Book

Chapters:

Next section

1. Recurrent Problems

Next section

2. Sums

Next section

3. Integer Functions

1. Floors and Ceilings
2. Floor/Ceiling Applications
3. Floor/Ceiling Recurrences
4. 'mod': The Binary Operation
5. Floor/Ceiling Sums

Next section

4. Number Theory

Next section

5. Binomial Coefficients

1. Basic Identities
2. Basic Practice
3. Tricks of the Trade
4. Generating Functions
5. Hypergeometric Functions
6. Hypergeometric Transformations
7. Partial Hypergeometric Sums
8. Mechanical Summation

Next section

6. Special Numbers

1. Stirling Numbers
2. Eulerian Numbers
3. Harmonic Numbers
4. Harmonic Summation
5. Bernoulli Numbers
6. Fibonacci Numbers
7. Continuants

Next section

7. Generating Functions

Next section

8. Discrete Probability

1. Definitions
2. Mean and Variance
3. Probability Generating Functions
4. Flipping Coins
5. Hashing

Next section

9. Asymptotics

Pedagogical dilemma: what to teach?

Chapters:

1. Recurrent Problems
2. Sums
3. Integer Functions
4. Number Theory
5. Binomial Coefficients
6. Special Numbers
7. Generating Functions
8. Discrete Probability
9. Asymptotics

About the choice: sums + recurrences + generating functions

Recurrent problems

Recurrent equation

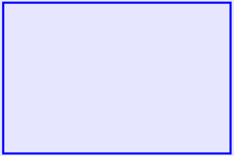
A number sequence $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ is called recurrent, if its general term together with k preceding terms satisfies a recurrent equation

$$a_n = g(a_{n-1}, \ldots, a_{n-k}),$$

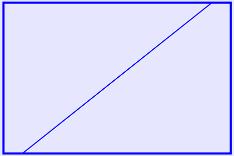
for every $n > k$, where the function $g : \mathbb{N}^k \longrightarrow \mathbb{N}$ (or $g : \mathbb{R}^k \longrightarrow \mathbb{R}$). The constant k is called order of the recurrent equation (or *difference equation*).

recurrent (<Latin *recurrere* – to run back) tagasipöörduv, taastuv / to run back.

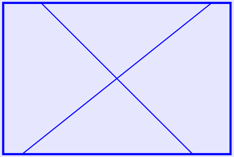
Regions of the plane defined by lines



$Q_0 = 1$



$Q_1 = 2$

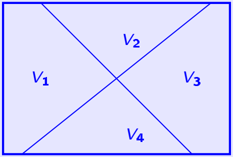


$Q_2 = 4$

In general: $Q_n = 2^n$?

Regions of the plane defined by lines

Actually ...



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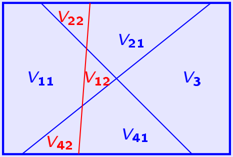

$$Q_3 = Q_2 + 3 = 7$$

Generally $Q_n = Q_{n-1} + n$.

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
Q_n	1	2	4	7	11	16	22	29	37	46	$\cdots$

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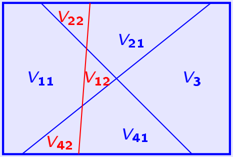

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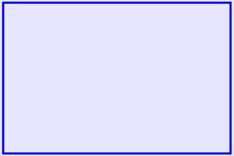
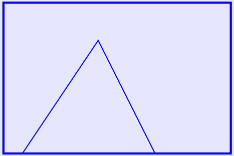
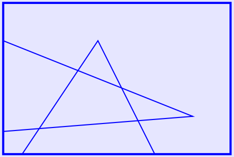


$T_0 = 1$



$T_1 = 2$

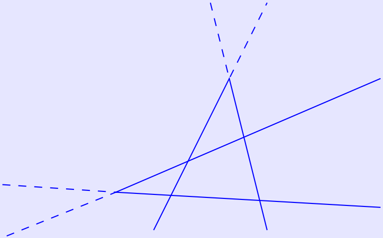


$T_2 = 7$

$T_3 = ?$

$T_n = ?$

Regions of the plane defined by lines



$$T_2 = Q_4 - 2\cdot 2 = 11 - 4 = 7$$
$$T_3 = Q_6 - 2\cdot 3 = 22 - 6 = 16$$
$$T_4 = Q_8 - 2\cdot 4 = 37 - 8 = 29$$
$$T_5 = Q_{10} - 2\cdot 5 = 56 - 10 = 46$$

$$T_n = Q_{2n} - 2n$$

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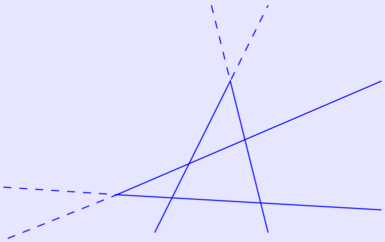

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Motivation: why to solve recursions?

- Computing by *closed form* is effective

$$P_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[1 + \frac{1}{12n} + \frac{1}{228n^2} + \frac{139}{51840n^3} + \mathscr{O}\left(\frac{1}{n^4}\right)\right]$$

- *Closed form* allows to analyse a function using "classical" techniques.

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- *Closed form* allows to analyse a function using "classical" techniques. For example: behaviour of *logistic map* depends on r:

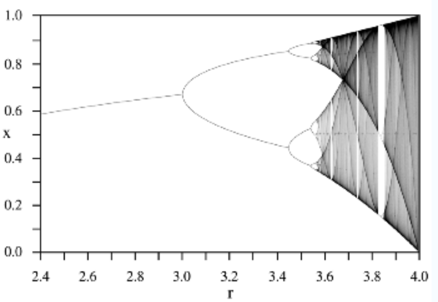
Recurrent equation:

$$x_{n+1} = rx_n(1-x_n)$$

Solution for $r=4$:

$$x_{n+1} = \sin^2(2^n\theta\pi)$$

where $\theta = \frac{1}{\pi}\sin^{-1}(\sqrt{x_0})$



ad hoc techniques: Guess and Confirm

Equation $f(n) = (n^2 - 1 + f(n-1))/2$, initial condition: $f(0) = 2$

- Let's compute some values:

n	0	1	2	3	4	5	6
$f(n)$	2	1	2	5	10	17	26

Guess: $f(n) = (n-1)^2 + 1$.

- Assuming that the guess holds for $n = k$, we prove that it holds in general by induction:

$$\begin{aligned} f(k+1) &= ((k+1)^2 - 1 + f(k))/2 = \\ &= (k^2 + 2k + (k-1)^2 + 1)/2 = \\ &= (k^2 + 2k + k^2 - 2k + 1 + 1)/2 = \\ &= (2k^2 + 2)/2 = k^2 + 1 \end{aligned}$$

QED.

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QED.

Solving recurrences

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n in terms of n.

"Algorithm"

1. Write down a single equation that expresses g_n in terms of other elements of the sequence. This equation should be valid for all integers n, assuming that $g_{-1} = g_{-2} = \cdots = 0$.
2. Multiply both sides of the equation by z^n and sum over all n. This gives, on the left, the sum $\sum_n g_n z^n$, which is the generating function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
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Example: Fibonacci numbers: equation $g_n = g_{n-1} + g_{n-2} + [n = 1]$

$$
\begin{aligned}
G(z) = \sum_n g_n z^n &= \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n = 1] z^n \\
&= \sum_n g_n z^{n+1} + \sum_n g_n z^{n+2} + z \\
&= z \sum_n g_n z^n + z^2 \sum_n g_n z^n + z \\
&= zG(z) + z^2 G(z) + z
\end{aligned}
$$

$$G(z) = \frac{z}{1 - z - z^2}$$

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$$
G(z) = \frac{z}{1 - z - z^2}
$$

Possible schedule

Chapters:

1. Recurrent Problems – weeks 1–3
2. Sums – weeks 4–6
3. Integer Functions – week 7
4. Number Theory
5. Binomial Coefficients
6. Special Numbers – weeks 8–9
7. Generating Functions – weeks 10–13
8. Discrete Probability
9. Asymptotics

Reserve – weeks 14 –16

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Contact

Instructors:	**Silvio Capobianco**	**Jaan Penjam**
Addresses:	`silvio@cs.ioc.ee`	`jaan@cs.ioc.ee`

Lectures and exercises: On Tuesdays at 14:00 – 17:30 in B126

Web page:

`http://cs.ioc.ee/cm/`