Number Theory

ITT9131 Konkreete Matemaatika

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Division (with remainder)

Definition

Let a and b be integers and $a > 0$. Then division of b by a is finding an integer quotient q and a remainder r satisfying the condition

$$b = aq + r \text{ , where } 0 \leqslant r < a.$$

Here

b	– dividend
a	– divider (=divisor) (=factor)
$q = \lfloor a/b \rfloor$	– quotient
$r = a \mod b$	– remainder (=residue)

Example

If $a = 3$ and $b = 17$, then

$$17 = 3 \cdot 5 + 2.$$

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Negative dividend

- If the divisor is positive, then the remainder is always **non-negative**.

For example

If $a = 3$ ja $b = -17$, then
$$-17 = 3 \cdot (-6) + 1.$$

- Integer b can be always represented as $b = aq + r$ with $0 \leqslant r < a$ due to the fact that b either coincides with a term of the sequence

$$\ldots, -3a, -2a, -a, 0, a, 2a, 3a, \ldots$$

or lies between two succeeding figures.

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NB! Division by a negative integer yields a negative remainder

$$5 \mod 3 = 5 - 3\lfloor 5/3 \rfloor = 2$$

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$$-5 \mod 3 = -5 - 3\lfloor -5/3 \rfloor = 1$$

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Be careful!

Some computer languages use another definition.

We assume $a > 0$ in further slides!

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Divisibility

Definition

Let a and b be integers. We say that a divides b , or a is a divisor of b, or b is a multiple of a, if there exists an integer m such that $b = a \cdot m$.

Notations:

- $a|b$ a divides b
- $a \setminus b$ a divides b
- $b \vdots a$ b is a multiple of a

For example

$$3|111 \qquad 7|-91 \qquad -7|-91$$

Divisors

Definitsioon

If $a|b$, then

- an integer a is called divisor or factor or multiplier of an integer b.

Properties

- Any integer b at least four divisors: $1, -1, b, -b$.
- $a|0$ for any integer a; reverse relation $0|a$ is valid only for $a = 0$. That means $0|0$.
- $1|b$ for any integer b,whereas $b|1$ is valid iff $b = 1$ or $b = -1$.

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More properties:

1. If $a|b$, then $\pm a|\pm b$.
2. If $a|b$ and $a|c$, for every m,n integer it is valid that $a|mb+nc$.
3. $a|b$ iff $ac|bc$ for every integer c.

The first property allows to restrict ourselves to study divisibility on positive integers.

It follows from the second property that if an integer a is a divisor of b and c, then it is the divisor their sum and difference.
Here a is called common divisor of b and c (as well as of $b+c$, $b-c$, $b+2c$ etc.)

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Greatest Common Divisor

Definition

The greatest common divisor (*gcd*) of two or more non-zero integers is the largest positive integer that divides the numbers without a remainder.

Example

The common divisors of 36 and 60 are 1, 2, 3, 4, 6, 12.
The greatest common divisor $gcd(36, 60) = 12$.

- The greatest common divisor exists always because of the set of common divisors of the given integers is non-empty and finite.

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The Euclidean algorithm

The algorithm to compute $gcd(a, b)$ for positive integers a and b

Input: Positive integers a and b, assume that $a > b$
Output: $gcd(a, b)$

- **while** $b > 0$
  do
  1. $r := a$ **mod** b
  2. $a := b$
  3. $b := r$

  od
- **return**(a)

Example: compute $gcd(2322, 654)$

a	b
2322	654
654	360
360	294
294	66
66	30
30	6
6	0

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Important questions to answer:

- Does the algorithm terminate for every input?
- Is the result the *greatest* common divisor?
- How long does it take?

Termination of the Euclidean algorithm

- In any cycle, the pair of integers (a, b) is replaced by (b, r), where r is the remainder of division of a by b.
- Hence $r < b$.
- The second number of the pair decreases, but remains non-negative, so the process cannot last infinitely long.

Correctness of the Euclidean algorithm

Theorem

If r is a remainder of division of a by b, then

$$gcd(a,b) = gcd(b,r)$$

Proof. It follows from the equality $a = bq + r$ that

1. if $d|a$ and $d|b$, then $d|r$
2. if $d|b$ and $d|r$, then $d|a$

In other words, the set of common divisors of a and b equals to the set of common divisors of b and r, recomputing of (b,r) does not change the greatest common divisor of the pair.

The number returned $r = gcd(r,0)$. Q.E.D.

Complexity of the Euclidean algorithm

Theorem

The number of steps of the Euclidean algorithm applied to two positive integers a and b is at most

$$1 + \log_2 a + \log_2 b.$$

Proof. Let consider the step where the pair (a, b) is replaced by (b, r). Then we have $r < b$ and $b + r \leqslant a$. Hence $2r < r + b \leqslant a$ or $br < ab/2$. This is that the product of the elements of the pair decreases at least 2 times.

If after k cycles the product is still positive, then $ab/2^k > 1$, that gives

$$k \leqslant \log_2(ab) = \log_2 a + \log_2 b$$

Q.E.D.

The numbers produced by the Euclidean algorithm

$$
\begin{aligned}
a &= bq_1 + r_1 & r_1 \text{ can be expressed in terms of } b \text{ and } a \\
b &= r_1 q_2 + r_2 & r_2 \text{ can be expressed in terms of } r_1 \text{ and } b \\
r_1 &= r_2 q_3 + r_3 & r_3 \text{ can be expressed in terms of } r_2 \text{ and } r_1 \\
&\dots\dots\dots\dots & \dots\dots\dots\dots \\
r_{k-3} &= r_{k-2} q_{k-1} + r_{k-1} & r_{k-1} \text{ can be expressed in terms of } r_{k-2} \text{ and } r_{k-3} \\
r_{k-2} &= r_{k-1} q_k + r_k & r_k \text{ can be expressed in terms of } r_{k-1} \text{ and } r_{k-2} \\
r_{k-1} &= r_k q_{k+1}
\end{aligned}
$$

Now, one can extract $r_k = gcd(a, b)$ from the second last equality and substitute there step-by-step $r_{k-1}, r_{k-2}, \dots$ using previous equations. We obtain finally that r_k equals to a linear combination of a and b with (not necessarily positive) integer coefficients.

GCD as a linear combination

Theorem (Bézout's identity)

Let $d = gcd(a,b)$. Then d can be written in the form

$$d = as + bt$$

where s and t are integers. In addition,

$$gcd(a,b) = \min\{n \geq 1 \mid \exists s,t \in \mathbb{Z} : n = as + bt\}.$$

For example: $a = 360$ and $b = 294$

$$gcd(a,b) = 294 \cdot (-11) + 360 \cdot 9 = -11a + 9b$$

Application of EA: solving of linear Diophantine Equations

Corollary

Let a, b and c be positive integers. The equation

$$ax + by = c$$

has integer solutions if and only if c is a multiple of $gcd(a, b)$.

The method: Making use of Euclidean algorithm, compute such coefficients s and t that $sa + tb = gcd(a, b)$. Then

$$x = \frac{cs}{gcd(a, b)}$$

$$y = \frac{ct}{gcd(a, b)}$$

Linear Diophantine Equations (2)

Example: $92x+17y=3$

From EA:

a	b	Seos
92	17	
17	7	$92=5\cdot 17+7$
7	3	$17=2\cdot 7+3$
3	1	$7=2\cdot 3+1$
1	0	

Transformations:

$$\begin{aligned}1 &= 7-2\cdot 3\\ &= 7-2\cdot(17-7\cdot 2)=(-2)\cdot 17+5\cdot 7=\\ &= (-2)\cdot 17+5\cdot(92-5\cdot 17)=5\cdot 92+(-27)\cdot 17\end{aligned}$$

$gcd(92,7)|3$ yields a solution

$$x=\frac{3\cdot 5}{gcd(92,17)}=3\cdot 5=15$$

$$y=\frac{3\cdot(-27)}{gcd(92,17)}=-3\cdot 27=-81$$

Linear Diophantine Equations (3)

Example: $5x+3y=2$ $\rightarrow$ many solutions

$$gcd(5,3)=1$$

As $1=2\cdot 5+3\cdot 3$, then one solution is:

$$x=2\cdot 2=4$$
$$y=-3\cdot 2=-6$$

As $1=(-10)\cdot 5+17\cdot 3$, then another solution is:

$$x=-10\cdot 2=-20$$
$$y=17\cdot 2=34$$

Example: $15x+9y=8$ $\rightarrow$ no solutions

Whereas, $gcd(15,9)=3$, then the equation can be expressed as

$$3\cdot(5x+3y)=8.$$

The left-hand side of the equation is divisible by 3, but the right-hand side is not, therefore the equality cannot be valid for any integer x and y.

Linear Diophantine Equations (3)

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$$gcd(5,3) = 1$$

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The left-hand side of the equation is divisible by 3, but the right-hand side is not, therefore the equality cannot be valid for any integer x and y.

More about Linear Diophantine Equations (1)

- General solution of a Diophantine equation $ax + by = c$ is

$$\begin{cases} x &= x_0 + \frac{kb}{gcd(a,b)} \\ y &= y_0 - \frac{ka}{gcd(a,b)} \end{cases}$$

where x_0 and y_0 are particular solutions and k is an integer.

- Particular solutions can be found by means of Euclidean algorithm:

$$\begin{cases} x_0 &= \frac{cs}{gcd(a,b)} \\ y_0 &= \frac{ct}{gcd(a,b)} \end{cases}$$

- This equation has a solution (where x and y are integers) if and only if $gcd(a,b)|c$
- The general solution above provides all integer solutions of the equation (see proof in http://en.wikipedia.org/wiki/Diophantine_equation)

More about Linear Diophantine Equations (2)

Example: $5x + 3y = 2$

We have found, that $gcd(5,3) = 1$ and its particular solutions are $x_0 = 4$ and $y_0 = -6$.

Thus, for any $k \in \mathbb{Z}$:

$$\begin{cases} x &= 4 + 3k \\ y &= -6 - 5k \end{cases}$$

Solutions of the equation for $k = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$ are infinite sequences of numbers:

$$\begin{array}{ccccccccccc} x & = & \dots, & -5, & -2, & 1, & 4, & 7, & 10, & 13, & \dots \\ y & = & \dots, & 9, & 4, & -1, & -6, & -11, & -16, & -21, & \dots \end{array}$$

Among others, if $k = -8$, then we get the solution $x = -20$ ja $y = 34$.

Next section

Prime and composite numbers

Every integer greater than 1 is either prime or composite, but not both:

- A positive integer p is called prime if it has just two divisors, namely 1 and p. By convention, 1 is not prime

Prime numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots$

- An integer that has three or more divisors is called composite

Composite numbers: $4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, \ldots$

Next subsection

Another application of EA

The Fundamental Theorem of Arithmetic

Every positive integer n can be written uniquely as a product of primes:

$$n = p_1 \dots p_m = \prod_{k=1}^{m} p_k, \qquad p_1 \leqslant \dots \leqslant p_m$$

Proof. Suppose we have two factorizations into primes

$$n = p_1 \dots p_m = q_1 \dots q_k, \qquad p_1 \leqslant \dots \leqslant p_m \text{ and } q_1 \leqslant \dots \leqslant q_k$$

Assume that $p_1 < q_1$. Since p_1 and q_1 are primes, $gcd(p_1, q_1) = 1$. That means that EA defines integers s and t that $sp_1 + tq_1 = 1$. Therefore

$$sp_1 q_2 \dots q_k + tq_1 q_2 \dots q_k = q_2 \dots q_k$$

Now p_1 divides both terms on the left, thus $q_2 \dots q_k / p_1$ is integer that contradicts with $p_1 < q_1$. This means that $p_1 = q_1$.

Similarly, using induction we can prove that $p_2 = q_2$, $p_3 = q_3$, etc

Q.E.D.

Canonical form of integers

- Every positive integer n can be represented uniquely as a product

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_p p^{n_p}, \qquad \text{where each } n_p \geqslant 0$$

For example:

$$\begin{aligned} 600 &= 2^3 \cdot 3^1 \cdot 5^2 \cdot 7^0 \cdot 11^0 \cdots \\ 35 &= 2^0 \cdot 3^0 \cdot 5^1 \cdot 7^1 \cdot 11^0 \cdots \\ 5\ 251\ 400 &= 2^3 \cdot 3^0 \cdot 5^2 \cdot 7^1 \cdot 11^2 \cdot 13^0 \cdot \cdots 29^0 \cdot 31^1 \cdot 37^0 \cdots \end{aligned}$$

Prime-exponent representation of integers

- Canonical form of an integer $n = \prod_p p^{n_p}$ provides a sequence of powers $\langle n_1, n_2, \ldots \rangle$ as another representation.

For example:

$$\begin{aligned} 600 &= \langle 3,1,2,0,0,0,\ldots \rangle \\ 35 &= \langle 0,0,1,1,0,0,0,\ldots \rangle \\ 5\ 251\ 400 &= \langle 3,0,2,1,2,0,0,0,0,0,1,0,0,\ldots \rangle \end{aligned}$$

Prime-exponent representation and arithmetic operations

Multiplication

Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} = \prod_p p^{m_p}$$

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_p p^{n_p}$$

Then

$$mn = p_1^{m_1+n_1} p_2^{m_2+n_2} \cdots p_k^{m_k+n_k} = \prod_p p^{m_p+n_p}$$

Using prime-exponent representation:

$$mn = \langle m_1 + n_1, m_2 + n_2, m_3 + n_3, \ldots \rangle$$

For example

$$\begin{aligned} 600 \cdot 35 &= \langle 3,1,2,0,0,0,\ldots \rangle \cdot \langle 0,0,1,1,0,0,0,\ldots \rangle \\ &= \langle 3+0, 1+0, 2+1, 0+1, 0+0, 0+0, \ldots \rangle \\ &= \langle 3,1,3,1,0,0,\ldots \rangle = 21\ 000 \end{aligned}$$

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Some other operations

The greatest common divisor and the least common multiple (*lcm*)

$$gcd(m,n) = \langle \min(m_1,n_1), \min(m_2,n_2), \min(m_3,n_3), \ldots \rangle$$

$$lcm(m,n) = \langle \max(m_1,n_1), \max(m_2,n_2), \max(m_3,n_3), \ldots \rangle$$

Example

$$120 = 2^3 \cdot 3^1 \cdot 5^1 = \langle 3,1,1,0,0,\cdots \rangle$$

$$36 = 2^2 \cdot 3^2 = \langle 2,2,0,0,\cdots \rangle$$

$$gcd(120,36) = 2^{\min(3,2)} \cdot 3^{\min(1,2)} \cdot 5^{\min(1,0)} = 2^2 \cdot 3^1 = \langle 2,1,0,0,\ldots \rangle = 12$$

$$lcm(120,36) = 2^{\max(3,2)} \cdot 3^{\max(1,2)} \cdot 5^{\max(1,0)} = 2^3 \cdot 3^2 \cdot 5^1 = \langle 3,2,1,0,0,\ldots \rangle = 360$$

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Properties of the GCD

Homogeneity

$gcd(na, nb) = n \cdot gcd(a, b)$ for every positive integer n.

Proof.

Let $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $b = p_1^{\beta_1} \cdots p_k^{\beta_k}$, and $gcd(a, b) = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, where $\gamma_i = min(\alpha_i, \beta_i)$. If $n = p_1^{n_1} \cdots p_k^{n_k}$, then

$$
\begin{aligned}
gcd(na, nb) &= p_1^{min(\alpha_1+n_1, \beta_1+n_1)} \cdots p_k^{min(\alpha_k+n_k, \beta_k+n_k)} = \\
&= p_1^{min(\alpha_1, \beta_1)} p_1^{n_1} \cdots p_k^{min(\alpha_k, \beta_k)} p_k^{n_k} = \\
&= p_1^{n_1} \cdots p_k^{n_k} p_1^{\gamma_1} \cdots p_k^{\gamma_k} = n \cdot gcd(a, b)
\end{aligned}
$$

Q.E.D.

Properties of the GCD

GCD and LCM

$gcd(a, b) \cdot lcm(a, b) = ab$ for every two positive integers a and b

Proof.

$$
\begin{aligned}
gcd(a, b) \cdot lcm(a, b) &= p_1^{\min(\alpha_1, \beta_1)} \cdots p_k^{\min(\alpha_k, \beta_k)} \cdot p_1^{\max(\alpha_1, \beta_1)} \cdots p_k^{\max(\alpha_k, \beta_k)} = \\
&= p_1^{\min(\alpha_1, \beta_1) + \max(\alpha_1, \beta_1)} \cdots p_k^{\min(\alpha_k, \beta_k) + \max(\alpha_k, \beta_k)} = \\
&= p_1^{\alpha_1 + \beta_1} \cdots p_k^{\alpha_k + \beta_k} = ab
\end{aligned}
$$

Q.E.D.

Relatively prime numbers

Definition

Two integers a and b are said to be relatively prime (or co-prime) if the only positive integer that evenly divides both of them is 1.

Notations used:

- $gcd(a, b) = 1$
- $a \perp b$

For example

$16 \perp 25$ and $99 \perp 100$

Some simple properties:

- Dividing a and b by their greatest common divisor yields relatively primes:

$$gcd\left(\frac{a}{gcd(a,b)}, \frac{b}{gcd(a,b)}\right) = 1$$

- Any two positive integers a and b can be represented as $a = a'd$ and $b = b'd$, where $d = gcd(a, b)$ and $a' \perp b'$

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Properties of relatively prime numbers

Theorem

If $a \perp b$, then $gcd(ac, b) = gcd(c, b)$ for every positive integer c.

Proof.

Assuming canonic representation of $a = \prod_p p^{\alpha_p}$, $b = \prod_p p^{\beta_p}$ and $c = \prod_p p^{\gamma_p}$, one can conclude that for any prime p:

- The premise $a \perp b$ implies that $p^{\min(\alpha_p, \beta_p)} = 1$, it is that either $\alpha_p = 0$ or $\beta_p = 0$.
- If $\alpha_p = 0$, then $p^{\min(\alpha_p + \gamma_p, \beta_p)} = p^{\min(\gamma_p, \beta_p)}$.
- If $\beta_p = 0$, then $p^{\min(\alpha_p + \gamma_p, \beta_p)} = p^{\min(\alpha_p + \gamma_p, 0)} = 1 = p^{\min(\gamma_p, 0)} = p^{\min(\gamma_p, \beta_p)}$.

Hence, the set of common divisors of ac and b is equal to the set of common divisors of c and b.

Q.E.D.

Divisibility

Observation

Let

$$a = \prod_p p^{\alpha_p}$$

and

$$b = \prod_p p^{\beta_p}.$$

Then $a|b$ iff $\alpha_p \leqslant \beta_p$ for every prime p.

Consequences from the theorems above

1. If $a \perp c$ and $b \perp c$, then $ab \perp c$
2. If $a|bc$ and $a \perp b$, then $a|c$
3. If $a|c$, $b|c$ and $a \perp b$, then $ab|c$

Example: compute $gcd(560, 315)$

$$\begin{aligned} gcd(560, 315) &= gcd(5 \cdot 112, 5 \cdot 63) = \\ &= 5 \cdot gcd(112, 63) = \\ &= 5 \cdot gcd(2^4 \cdot 7, 63) = \\ &= 5 \cdot gcd(7, 63) \\ &= 5 \cdot 7 = 35 \end{aligned}$$

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The number of divisors

- Canonic form of a positive integer permits to compute the number of its factors without factorization:
- If
$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$
then any divisor of n can be constructed by multiplying $0, 1, \cdots, n_1$ times the prime divisor p_1, then $0, 1, \cdots, n_2$ times the prime divisor p_2 etc.
- Then the number of divisors of n should be
$$(n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

Example

Integer 694 575 has $694\,575 = 3^4 \cdot 5^2 \cdot 7^3$ on $(4+1)(2+1)(3+1) = 60$ factors.

Next subsection

Number of primes

Euclid's theorem

There are infinitely many prime numbers.

Proof. Let's assume that there is finite number of primes:

$$p_1, p_2, p_3, \dots, p_k.$$

Consider

$$n = p_1 p_2 p_3 \cdots p_k + 1.$$

Like any other natural number, n is divisible at least by 1 and itself, i.e. it can be prime. Dividing n by $p_1, p_2, p_3, \dots$ or p_k yields the remainder 1. So, n should be prime that differs from any of numbers $p_1, p_2, p_3, \dots, p_k$, that leads to a contradiction with the assumption that the set of primes is finite.

Q.E.D.

Number of primes (another proof)

Theorem

There are infinitely many prime numbers.

Proof. For any natural number n, there exits a prime number greater than n:

Let p be the smallest divisor of $n! + 1$ that is greater than 1. Then

- p is a prime number, as otherwise it wouldn't be the smallest divisor.
- $p > n$, as otherwise $p | n!$ and $p | n! + 1$ and $p | (n! + 1) - n! = p | 1$.

Q.E.D.

Number of primes: A proof by Paul Erdős

Theorem

$$\sum_{p\,\mathrm{prime}} \frac{1}{p} = \infty$$

Number of primes: A proof by Paul Erdős

Theorem

$$\sum_{p\,\text{prime}} \frac{1}{p} = \infty$$

By contradiction, assume $\sum_{p\,\text{prime}} \frac{1}{p} < \infty$.

- Call a prime p *large* if $\sum_{q\,\text{prime} \geq p} \frac{1}{q} < \frac{1}{2}$. Let N be the number of *small* primes.
- For $m \geq 1$ let $U_m = \{1 \leq n \leq m \mid n\,\text{only has small prime factors}\}$.
  For $n \in U_m$ it is $n = d \cdot k^2$ where q is a product of distinct small primes. Then
  $$|U_m| \leq 2^N \cdot \sqrt{m}$$
- For $m \geq 1$ and p prime let $D_{m,p} = \{1 \leq n \leq m \mid p \setminus n\}$. Then
  $$|\{1, \ldots, m\} \setminus U_m| \leq \sum_{p\,\text{large prime}} |D_{m,p}| < \frac{m}{2}$$
- Then $\frac{m}{2} \leq |U_m| \leq 2^N \sqrt{m}$: this is false for m large enough.

Primes are distributed "very irregularly"

- Since all primes except 2 are odd, the difference between two primes must be at least two, except 2 and 3.
- Two primes whose difference is two are called twin primes. For example (17, 19) or (3557 and 3559). There is **no proof** of the hypothesis that there are infinitely many twin primes.

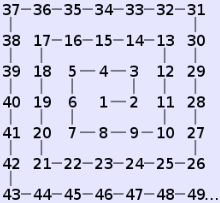
Theorem

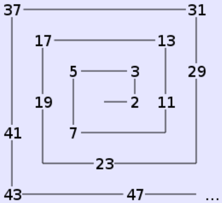
For every positive integer k, there exist k consecutive composite integers.

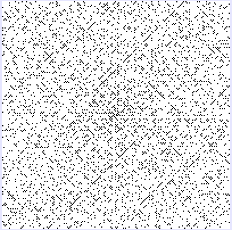
Proof. Let $n = k + 1$ and consider the numbers $n! + 2, n! + 3, \ldots, n! + n$. All these numbers are composite because of $i | n! + i$ for every $i = 2, 3, \ldots, n$.

Q.E.D.

Distribution diagrams for primes







The prime counting function $\pi(n)$

- Definition:

$$\pi(n) = \text{number of primes in the set} \{1, 2, \ldots, n\}$$

- The first values:

$$\begin{array}{llll} \pi(1) = 0 & \pi(2) = 1 & \pi(3) = 2 & \pi(4) = 2 \\ \pi(5) = 3 & \pi(6) = 3 & \pi(7) = 4 & \pi(8) = 4 \end{array}$$

The Prime Number Theorem

Theorem

The quotient of division of $\pi(n)$ by $n/\ln n$ will be arbitrarily close to 1 as n gets large. It is also denoted as

$$\pi(n) \sim \frac{n}{\ln n}$$

- Studying prime tables C. F. Gauss come up with the formula in $\sim$1791.
- J. Hadamard and C. de la Vallée Poussin proved the theorem independently from each other in 1896.

The Prime Number Theorem (2)

Example: How many primes are with 200 digits?

- The total number of positive integers with 200 digits:

$$10^{200} - 10^{199} = 9 \cdot 10^{199}$$

- Approximate number of primes with 200 digits

$$\pi(10^{200}) - \pi(10^{199}) \approx \frac{10^{200}}{200 \ln 10} - \frac{10^{199}}{199 \ln 10} \approx 1,95 \cdot 10^{197}$$

- Percentage of primes

$$\frac{1,95 \cdot 10^{197}}{9 \cdot 10^{199}} \approx \frac{1}{460} = 0.22\%$$

Warmup: Extending $\pi(x)$ to positive reals

Problem

Let $\pi(x)$ be the number of primes which are not larger than $x \in \mathbb{R}$.
Prove or disprove: $\pi(x) - \pi(x-1) = [x\,\mathrm{is\,prime}]$.

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Solution

The formula is true if x is integer: but x is real ...

But clearly $\pi(x) = \pi(\lfloor x \rfloor)$: then

$$\begin{aligned}
\pi(x) - \pi(x-1) &= \pi(\lfloor x \rfloor) - \pi(\lfloor x-1 \rfloor) \\
&= \pi(\lfloor x \rfloor) - \pi(\lfloor x \rfloor - 1) \\
&= [\lfloor x \rfloor \ \mathrm{is\,prime}] \ ,
\end{aligned}$$

which is true.