Recurrent Problems

ITT9131 Konkreetne Matemaatika

Chapter One

The Tower of Hanoi

Lines in the Plane

The Josephus Problem

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2 Generalization of Josephus function

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4 Binary representation of generalized Josephus function

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4 Binary representation of generalized Josephus function

Binary expansion of $n = 2^m + \ell$

Denote

$$n = (b_m b_{m-1} \dots b_1 b_0)_2,$$

where $b_i \in \{0, 1\}$ and $b_m = 1$.

This notation stands for

$$n = b_m 2^m + b_{m-1} 2^{m-1} + \dots b_1 2 + b_0$$

For example

$$20 = (10100)_2 \qquad \text{and} \qquad 83 = (1010011)_2$$

Binary expansion of $n = 2^m + \ell$, where $0 \leqslant \ell < 2^m$

Observations:

1. $\ell = (0 b_{m-1} \ldots b_1 b_0)_2$
2. $2\ell = (b_{m-1} \ldots b_1 b_0 0)_2$
3. $2^m = (10 \ldots 00)_2$ and $1 = (00 \ldots 01)_2$
4. $n = 2^m + \ell = (1 b_{m-1} \ldots b_1 b_0)_2$
5. $2\ell + 1 = (b_{m-1} \ldots b_1 b_0 1)_2$

Corollary

$$J((\boxed{1}\, b_{m-1} \ldots b_1 b_0)_2) = (b_{m-1} \ldots b_1 b_0 \,\boxed{1})_2$$

shift

Binary expansion of $n = 2^m + \ell$, where $0 \leqslant \ell < 2^m$

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Binary expansion of $n = 2^m + \ell$, where $0 \leqslant \ell < 2^m$

Example

$$100 = 64 + 32 + 4$$
$$J(100) = J((1100100)_2) = (1001001)_2$$
$$J(100) = 64 + 8 + 1 = 73$$

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1 Binary representation

2 Generalization of Josephus function

3 Intermezzo: The repertoire method

4 Binary representation of generalized Josephus function

Generalization

Josephus function $J : \mathbb{N} \longrightarrow \mathbb{N}$

was defined using recurrences:

$$\begin{aligned} J(1) &= 1; && \\ J(2n) &= 2J(n) - 1, && \text{for } n \geqslant 1; \\ J(2n+1) &= 2J(n) + 1, && \text{for } n \geqslant 1. \end{aligned}$$

Introducing natural constants α, β and γ, generalize it as follows:

$$\begin{aligned} f(1) &= \alpha; && \\ f(2n) &= 2f(n) + \beta, && \text{for } n \geqslant 1; \\ f(2n+1) &= 2f(n) + \gamma, && \text{for } n \geqslant 1. \end{aligned}$$

$$J = f\ , \qquad \text{for } \alpha = 1, \beta = -1, \gamma = 1$$

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Repertoire method

To find closed form of a function f:

- **Step 1** Find few initial values for f;
- **Step 2** Find (or guess) closed formula from the values found by Step 1
  (*examine a repertoire of cases and combine them to find general closed formula*)
- **Step 3** Verify the closed formula constructed as the result of Step 2.

The idea is to examine a repertoire of cases and use it to find a general closed formula for the recurrently defined function.

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Repertoire method for generalized f-n f: STEP 1

n	$f(n)$	Calculation
1	α	$f(1) = \alpha$
2	$2\alpha + \beta$	$f(2) = 2f(1) + \beta$
3	$2\alpha + \gamma$	$f(3) = 2f(1) + \gamma$
4	$4\alpha + 3\beta$	$f(4) = 2f(2) + \beta$
5	$4\alpha + 2\beta + \gamma$	$f(5) = 2f(2) + \gamma$
6	$4\alpha + \beta + 2\gamma$	$f(6) = 2f(3) + \beta$
7	$4\alpha + 3\gamma$	$f(7) = 2f(3) + \gamma$
8	$8\alpha + 7\beta$	$f(8) = 2f(4) + \beta$
9	$8\alpha + 6\beta + \gamma$	$f(9) = 2f(4) + \gamma$

Repertoire method for generalized f-n f: STEP 2

Observations:

For n=1, 2, . . . , 9, taking $n = 2^k + \ell$

- Coefficient of α is 2^k;
- Coefficient of β is $2^k - 1 - \ell$;
- Coefficient of γ is ℓ;

Repertoire method for generalized f-n f: STEP 3

Proposition:

If the function f is given by recurrences

$$\begin{aligned} f(1) &= \alpha; \\ f(2n) &= 2f(n) + \beta, && \text{for } n \geqslant 1; \\ f(2n+1) &= 2f(n) + \gamma, && \text{for } n \geqslant 1, \end{aligned}$$

and let $n = 2^k + \ell$, then

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma,$$

where

$$\begin{aligned} A(n) &= 2^k; \\ B(n) &= 2^k - 1 - \ell; \\ C(n) &= \ell. \end{aligned}$$

Proof of the proposition (1)

Lemma 1. $A(n) = 2^k$, where $n = 2^k + \ell$ and $0 \leqslant \ell < 2^k$.

Proof.

Let $\alpha = 1$ and $\beta = \gamma = 0$. Then $f(n) = A(n)$ and

$$A(1) = 1 \; ; \qquad A(2n) = 2A(n) \text{ , for } n > 0 \; ; \qquad A(2n+1) = 2A(n) \text{ , for } n > 0$$

Proof by induction over k:

Basis: If $k = 0$, then $n = 2^0 + \ell$ and $0 \leqslant \ell < 1$. Thus n=1 and

$$A(1) = 2^0 = 1$$

Step: Let assume that $A(2^{k-1} + t) = 2^{k-1}$, where $0 \leqslant t < 2^{k-1}$ Two cases:

- If n is even, then ℓ is even and $\ell/2 < 2^{k-1}$, thus

$$A(n) = A(2^k + \ell) = 2A(2^{k-1} + \ell/2) = 2 \cdot 2^{k-1} = 2^k$$

- If n is odd, then $\ell - 1$ is even and $(\ell - 1)/2 < 2^{k-1}$, thus

$$A(n) = A(2^k + \ell) = 2A(2^{k-1} + (\ell - 1)/2) = 2 \cdot 2^{k-1} = 2^k$$

□

Proof of the proposition (2)

Lemma 2. $A(n) - B(n) - C(n) = 1$, for all $n \in \mathbb{N}$.

Proof.

Let f be the constant function $f(n) = 1$. Then

$$f(1) = \alpha \; ; \qquad f(2n) = 2f(n) + \beta \; ; \qquad f(2n+1) = 2f(n) + \gamma$$

or

$$1 = \alpha \; ; \qquad 1 = 2 + \beta \; ; \qquad 1 = 2 + \gamma.$$

As this must hold for every $n \geq 1$, it must be $\alpha = 1$ and $\beta = \gamma = -1$.

$\square$

Proof of the proposition (3)

Lemma 3. $A(n) + C(n) = n$, for all $n \in \mathbb{N}$.

Proof.

Let $f(n) = n$. Then

$$f(1) = \alpha \;; \qquad f(2n) = 2f(n) + \beta \;; \qquad f(2n+1) = 2f(n) + \gamma$$

or

$$1 = \alpha \;; \qquad 2n = 2n + \beta \;; \qquad 2n+1 = 2n + \gamma.$$

The solution is $\alpha = 1$, $\beta = 0$ and $\gamma = 1$.

□

Proof of the proposition (4)

From Lemma 3 and Lemma 1 we can conclude

$$2^k + C(n) = A(n) + C(n) = n = 2^k + \ell,$$

that gives

$$C(n) = \ell$$

From Lemma 2 follows

$$B(n) = A(n) - 1 - C(n) = 2^k - 1 - \ell$$

$\square$

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The repertoire method: Basic ideas

Consider the following recursion scheme:

$$\begin{aligned} g(0) &= \alpha\,, \\ g(n+1) &= \Phi(g(n)) + \Psi(n;\beta,\gamma,\dots) \quad \text{for } n \geq 0\,. \end{aligned}$$

Suppose that:

1. Φ is linear in g: if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$, then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
   No hypotheses are made on the dependence of g on n.
2. Ψ is linear in each of the $m-1$ parameters $\beta, \gamma, \dots$
   No hypotheses are made on the dependence of Ψ on n.

Then the whole system is linear in the parameters $\alpha, \beta, \gamma, \dots$
We can then look for a general solution of the form

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \dots$$

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The repertoire method: Description

Suppose we have a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfying the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$
2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then the functions $A(n), B(n), C(n), \ldots$ are uniquely determined. The reason is that, for every fixed n,

$$\begin{array}{llllll} \alpha_1 A(n) & +\beta_1 B(n) & +\gamma_1 C(n) & +\ldots & = & g_1(n) \\ \vdots & & & & = & \vdots \\ \alpha_m A(n) & +\beta_m B(n) & +\gamma_m C(n) & +\ldots & = & g_m(n) \end{array}$$

is a system of m linear equations in the m unknowns $A(n), B(n), C(n), \ldots$ whose coefficients matrix is invertible.

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Binary representation of generalized Josephus function

Definition

Generalized Josephus function (GJ-function) is defined for α, β_0, β_1 as follows:

$$f(1) = \alpha$$
$$f(2n+j) = 2f(n) + \beta_j \ , \qquad \text{for } j = 0, 1 \text{ and } n > 0.$$

We obtain the definition used before if to select $\beta_0 = \beta$ and $\beta_1 = \gamma$.

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Binary representation of generalized Josephus function (2)

Case A. Let's consider even number $2n$

If $2n = 2^m + \ell$, then the binary notation is

$$2n = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0, 1\}$, $b_0 = 0$ and $b_m = 1$.

Hence

$$n = b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

Binary representation of generalized Josephus function (3)

Case B. Odd number $2n+1$

If $2n+1 = 2^m + \ell$, then the binary notation is

$$2n+1 = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n+1 = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 1$ and $b_m = 1$.

We get

$$\begin{aligned} 2n+1 &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + 1 \\ 2n &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 \\ n &= b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1 \end{aligned}$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

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$$2n+1 = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 1$ and $b_m = 1$.

We get

$$\begin{aligned} 2n+1 &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + 1 \\ 2n &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 \\ n &= b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1 \end{aligned}$$

Same results for cases A and B indicates that we don't need to consider even and odd cases separately.

Binary representation of generalized Josephus function (4)

Let's evaluate:

$$\begin{aligned}
f((b_m, b_{m-1}, \dots, b_1, b_0)_2) &= 2f((b_m, b_{m-1}, \dots, b_1)_2) + \beta_{b_0} \\
&= 2(2f((b_m, b_{m-1}, \dots, b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, \dots, b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&\ \vdots \\
&= f((b_m)_2)2^m + \beta_{b_{m-1}} 2^{m-1} + \dots + \beta_{b_1} 2 + \beta_{b_0} \\
&= f(1)2^m + \beta_{b_{m-1}} 2^{m-1} + \dots + \beta_{b_1} 2 + \beta_{b_0} \\
&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \dots + \beta_{b_1} 2 + \beta_{b_0},
\end{aligned}$$

where

$$\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}$$

$$f((b_m b_{m-1} \dots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_1} \beta_{b_0})_2$$

Binary representation of generalized Josephus function (4)

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where

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\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}
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$$
f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2
$$

Example

Original Josephus function: $\alpha = 1,\ \beta_0 = -1,\ \beta_1 = 1$ i.e.

$$\begin{aligned} f(1) &= 1 \\ f(2n) &= 2f(n) - 1 \\ f(2n+1) &= 2f(n) + 1 \end{aligned}$$

Compute

$$f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2$$

$$\begin{aligned} f(100) = f((1100100)_2) &= (1, 1, -1, -1, 1, -1, -1)_2 \\ &= 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73 \end{aligned}$$

Nothing special with base 2!

Recursion by division by 3

$$\begin{aligned} f(1) &= \alpha_1 \\ f(2) &= \alpha_2 \\ f(3n) &= 2f(n) + \beta_0 \\ f(3n+1) &= 2f(n) + \beta_1 \\ f(3n+2) &= 2f(n) + \beta_2 \end{aligned}$$

Representation in base 3

$n = (t_m t_{m-1} \ldots t_1 t_0)_3 = t_m \cdot 3^m + t_{m-1} \cdot 3^{m-1} + \ldots + t_1 \cdot 3 + t_0$
where $t_1 \in \{0, 1, 2\}$ are *trits* (instead of bits)

Nothing special with base 2! (cont.)

Let's evaluate:

Let $n = (t_m t_{m-1} \dots t_1 t_0)_3$. Then $t_m \in \{1, 2\}$ and:

$$
\begin{aligned}
f((t_m t_{m-1} \dots t_1 t_0)_3) &= 2 \cdot f((t_m t_{m-1} \dots t_1)_3) + \beta_{t_0} \\
&= 2 \cdot (2 \cdot f((t_m t_{m-1} \dots t_2)_3) + \beta_{t_1}) + \beta_{t_0} \\
&= 4 f((t_m t_{m-1} \dots t_2)_3) + 2\beta_{t_1} + \beta_{t_0} \\
&= \dots \\
&= 2^m \cdot f(t_m) + 2^{m-1} \cdot \beta_{t_{m-1}} + \dots + 2 \cdot \beta_{t_1} + \beta_{t_0} \\
&= 2^m \cdot \alpha_{t_m} + 2^{m-1} \cdot \beta_{t_{m-1}} + \dots + 2 \cdot \beta_{t_1} + \beta_{t_0}
\end{aligned}
$$

Nothing special with base 2! (conclusion)

General problem, general solution

Let $d \geq 2$ be an integer. Consider the recurrent problem:

$$\begin{aligned} f(i) &= \alpha_i & \text{for } i \in \{1, \ldots, d-1\} \\ f(dn+j) &= cf(n) + \beta_j & \text{for } i \in \{0, \ldots, d-1\} \end{aligned}$$

Let $n = (x_m x_{m-1} \ldots x_1 x_0)_d$. Then $x_m \in \{1, \ldots, d-1\}$ and:

$$\begin{aligned} f((x_m x_{m-1} \ldots x_1 x_0)_d) &= c \cdot f((x_m x_{m-1} \ldots x_1)_d) + \beta_{x_0} \\ &= c \cdot (c \cdot f((x_m x_{m-1} \ldots x_2)_d) + \beta_{x_1}) + \beta_{x_0} \\ &= \ldots \\ &= c^m \cdot \alpha_{x_m} + c^{m-1} \cdot \beta_{x_{m-1}} + \ldots + c \cdot \beta_{x_1} + \beta_{x_0} \\ &= (\alpha_{x_m} \beta_{x_{m-1}} \ldots \beta_{x_1} \beta_{x_0})_c \end{aligned}$$