Special Numbers

ITT9131 Konkreetne Matemaatika

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Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ {n \atop k} \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Example: splitting a four-element set into two parts

$$\begin{array}{cccc} \{1,2,3\} \cup \{4\} & \{1,2,4\} \cup \{3\} & \{1,3,4\} \cup \{2\} & \{2,3,4\} \cup \{1\} \\ \{1,2\} \cup \{3,4\} & \{1,3\} \cup \{2,4\} & \{1,4\} \cup \{2,3\} & \end{array}$$

Hence $\left\{ {4 \atop 2} \right\} = 7$

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ {n \atop k} \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Some special cases: (1)

$k = 0$ We can partition a set into no nonempty parts if and only if the set is empty.

That is: $\left\{ {n \atop 0} \right\} = [n = 0]$.

$k = 1$ We can partition a set into one nonempty part if and only if the set is nonempty.

That is: $\left\{ {n \atop 1} \right\} = [n > 0]$.

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ {n \atop k} \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Some special cases: (2)

$k = n$ If $n > 0$, the only way to partition a set with n elements into n nonempty parts, is to put every element by itself.

That is: $\left\{ {n \atop n} \right\} = 1$. (This also matches the case $n = 0$.)

$k = n - 1$ Choosing a partition of a set with n elements into $n - 1$ nonempty subsets, is the same as choosing the two elements that go together.

That is: $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Some special cases (3)

$k = 2$ Let X be a set with two or more elements.

- Each partition of X into two subsets is identified by two ordered pairs $(A, X \setminus A)$ for $A \subseteq X$.
- There are 2^n such pairs, but $(\emptyset, X)$ and $(X, \emptyset)$ do not satisfy the nonemptiness condition.
- Then $\left\{ n \atop 2 \right\} = \dfrac{2^n - 2}{2} = 2^{n-1} - 1$ for $n \geqslant 2$.

In general, $\left\{ n \atop 2 \right\} = (2^{n-1} - 1)[n \geqslant 1]$

Stirling numbers of the second kind

Definition

The Stirling number of the second kind $\left\{ {n \atop k} \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty groups, then decide which group to add the nth element to.
2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty groups out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}, \qquad \text{for } n > 0,$$

that yields the following triangle:

Stirling's triangle for subsets

n	$\left\{ {n \atop 0} \right\}$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$	$\left\{ {n \atop 6} \right\}$	$\left\{ {n \atop 7} \right\}$	$\left\{ {n \atop 8} \right\}$	$\left\{ {n \atop 9} \right\}$
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

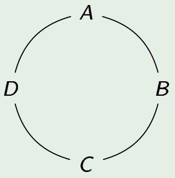
Next subsection

Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[\begin{matrix} n \\ k \end{matrix}\right]$, read "$n$ cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Circle is a cyclic arrangement



- Circle can be written as $[A, B, C, D]$;
- It means that $[A, B, C, D] = [B, C, D, A] = [C, D, A, B] = [D, A, B, C]$;
- It is not same as $[A, B, D, C]$ or $[D, C, B, A]$.

Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[{n \atop k}\right]$, read "n cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Example: splitting a four-element set into two circles

$$
\begin{array}{cccc}
[1,2,3]\ [4] & [1,2,4]\ [3] & [1,3,4]\ [2] & [2,3,4]\ [1] \\
[1,3,2]\ [4] & [1,4,2]\ [3] & [1,4,3]\ [2] & [2,4,3]\ [1] \\
[1,2]\ [3,4] & [1,3]\ [2,4] & [1,4]\ [2,3] &
\end{array}
$$

Hence $\left[{4 \atop 2}\right] = 11$

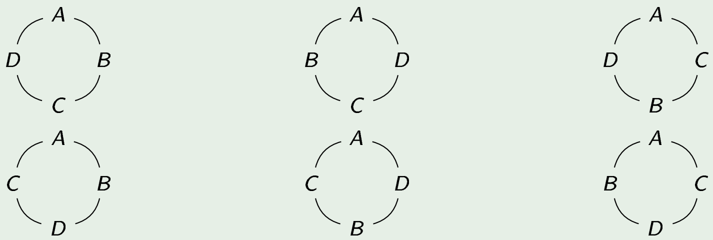
Stirling numbers of the first kind

Definition

The **Stirling number of the first kind** $\left[n \atop k \right]$, read "*n* cycle *k*", is the number of ways to partition of a set with n elements into k non-empty circles.

Some special cases (1):

$k = 1$ To arrange one circle of n objects: choose the order, and forget which element was the first. That is: $\left[n \atop 1 \right] = n!/n = (n-1)!$.



Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[{n \atop k} \right]$, read "$n$ cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Some special cases (2):

$k = n$ Every circle is the singleton and there is just one partition into circles. That is, $\left[{n \atop n} \right] = 1$ for any n:

$$[1] \quad [2] \quad [3] \quad [4]$$

$k = n-1$ The partition into circles consists of $n-2$ singletons and one pair. So $\left[{n \atop n-1} \right] = \binom{n}{2}$, the number of ways to choose a pair:

$$[1,2] \quad [3] \quad [4] \qquad [1,3] \quad [2] \quad [4] \qquad [1,4] \quad [2] \quad [3]$$

$$[2,3] \quad [1] \quad [4] \qquad [2,4] \quad [1] \quad [3] \qquad [3,4] \quad [1] \quad [2]$$

Stirling numbers of the first kind

Definition

The Stirling number of the first kind $\left[{n \atop k}\right]$, read "n cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty cycles, then decide which element to put the nth one *after*.
2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty cycles out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right], \qquad \text{for } n > 0,$$

that yields the following triangle:

Stirling's triangle for circles

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$	$\left[{n \atop 7}\right]$	$\left[{n \atop 8}\right]$	$\left[{n \atop 9}\right]$
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

Warmup: A closed formula for $\begin{bmatrix} n \\ 2 \end{bmatrix}$

Theorem

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\,H_{n-1}\,[n \geqslant 1]$$

Warmup: A closed formula for $\begin{bmatrix} n \\ 2 \end{bmatrix}$

Theorem

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1}\,[n \geqslant 1]$$

The formula is true for $n = 0$ and $n = 1$ ($H_0 = 0$ as an empty sum) so let $n \geqslant 2$.

- For $k = 1, \ldots, n-1$ there are $\binom{n}{k}$ ways of splitting n objects into a group of k and one of $n-k$. Each such way appears once for k, and once for $n-k$.
- To each splitting correspond $\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-k \\ 1 \end{bmatrix} = (k-1)!(n-k-1)!$ pairs of cycles.
- Then

$$\begin{aligned}
\begin{bmatrix} n \\ 2 \end{bmatrix} &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(k-1)!(n-k-1)! \\
&= \frac{n!}{2}\sum_{k=1}^{n-1}\frac{1}{k(n-k)} \\
&= \frac{n!}{2}\sum_{k=1}^{n-1}\frac{1}{n}\left(\frac{1}{k}+\frac{1}{n-k}\right) \\
&= (n-1)!H_{n-1}
\end{aligned}$$

Next subsection

1 Stirling numbers
- Stirling numbers of the second kind
- Stirling numbers of the first kind
- Basic Stirling number identities, for integer $n \geqslant 0$
- Extension of Stirling numbers

2 Fibonacci Numbers

Basic Stirling number identities, for integer $n \geqslant 0$

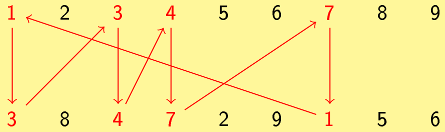
Some identities and inequalities we have already observed:

- $\left\{ {n \atop 0} \right\} = \left[{n \atop 0} \right] = [n = 0]$
- $\left\{ {n \atop 1} \right\} = [n > 0]$ and $\left[{n \atop 1} \right] = (n-1)![n > 0]$
- $\left\{ {n \atop 2} \right\} = (2^{n-1} - 1)[n > 0]$ and $\left[{n \atop 2} \right] = (n-1)!H_{n-1}[n > 0]$
- $\left\{ {n \atop n-1} \right\} = \left[{n \atop n-1} \right] = \binom{n}{2} = \dfrac{n(n-1)}{2}$
- $\left\{ {n \atop n} \right\} = \left[{n \atop n} \right] = \binom{n}{n} = 1$
- $\left\{ {n \atop k} \right\} = \left[{n \atop k} \right] = \binom{n}{k} = 0$, if $k > n$ or $k < 0$

Basic Stirling number identities (2)

For any integer $n \geqslant 0$, $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$

Permutations define cyclic arrangement and vice versa,
for example:



Thus the permutation $\pi = 384729156$ of $\{1,2,3,4,5,6,7,8,9\}$ is equivalent to the circle arrangement

$[1,3,4,7]$ $[2,8,5]$ $[6,9]$

Basic Stirling number identities (2)

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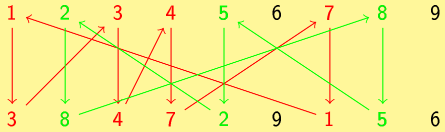

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Basic Stirling number identities (2)

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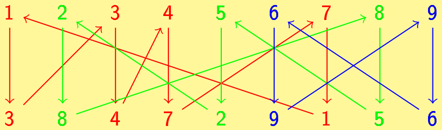

Thus the permutation $\pi = 384729156$ of $\{1,2,3,4,5,6,7,8,9\}$ is equivalent to the circle arrangement

$[1,3,4,7]$ $[2,8,5]$ $[6,9]$

Basic Stirling number identities (3)

Observation

$$x^0 = x^{\underline{0}}$$
$$x^1 = x^{\underline{1}}$$
$$x^2 = x^{\underline{1}} + x^{\underline{2}}$$
$$x^3 = x^{\underline{1}} + 3x^{\underline{2}} + x^{\underline{3}}$$
$$x^4 = x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + x^{\underline{4}}$$
$$\cdots\cdots$$

n	$\left\{ {n \atop 0} \right\}$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

Does the following general formula hold?

$$x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}$$

Basic Stirling number identities (3a)

Inductive proof of $x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}$

- Considering that $x^{\underline{k+1}} = x^{\underline{k}}(x-k)$ we obtain that $x \cdot x^{\underline{k}} = x^{\underline{k+1}} + kx^{\underline{k}}$
- Hence

$$
\begin{aligned}
x \cdot x^{n-1} = x \sum_k \left\{ {n-1 \atop k} \right\} x^{\underline{k}} &= \sum_k \left\{ {n-1 \atop k} \right\} x^{\underline{k+1}} + \sum_k \left\{ {n-1 \atop k} \right\} kx^{\underline{k}} \\
&= \sum_k \left\{ {n-1 \atop k-1} \right\} x^{\underline{k}} + \sum_k \left\{ {n-1 \atop k} \right\} kx^{\underline{k}} \\
&= \sum_k \left(\left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \right) x^{\underline{k}} = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}
\end{aligned}
$$

Q.E.D.

Basic Stirling number identities (4)

Observation

$$x^{\overline{0}} = x^0$$

$$x^{\overline{1}} = x^1$$

$$x^{\overline{2}} = x^1 + x^2$$

$$x^{\overline{3}} = 2x^1 + 3x^2 + x^3$$

$$x^{\overline{4}} = 6x^1 + 11x^2 + 6x^3 + x^4$$

$$\ldots\ldots$$

Generating function for Stirling cycle numbers:

$$x^{\overline{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k, \qquad \text{for } n \geqslant 0$$

Basic Stirling number identities (4a)

Generating function of the Stirling numbers of the first kind

$$\sum_k \left[{n \atop k}\right] z^k = z^{\overline{n}} \ \forall n \geqslant 0$$

The formula is clearly true for $n = 0$ and $n = 1$.
If it is true for $n - 1$, then:

$$\begin{aligned}
z^{\overline{n}} &= z^{\overline{n-1}}(z+n-1) \\
&= \left(\sum_k \left[{n-1 \atop k}\right] z^k\right)(z+n-1) \\
&= \sum_k \left[{n-1 \atop k}\right] z^{k+1} + (n-1)\sum_k \left[{n-1 \atop k}\right] z^k \\
&= \sum_k \left[{n-1 \atop k-1}\right] z^k + (n-1)\sum_k \left[{n-1 \atop k}\right] z^k \\
&= \sum_k \left((n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]\right) z^k,
\end{aligned}$$

whence the thesis.

Basic Stirling number identities (5)

Reversing the formulas for falling and rising factorials

For every $n \geqslant 0$,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} \text{ and } x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

Basic Stirling number identities (5)

Reversing the formulas for falling and rising factorials

For every $n \geqslant 0$,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} \text{ and } x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

Proof

As $x^{\underline{k}} = (-1)^k (-x)^{\overline{k}}$, we can rewrite the known equalities as:

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^k (-x)^{\overline{k}} \text{ and } (-1)^n (-x)^{\underline{n}} = \sum_k \left[{n \atop k} \right] x^k$$

But clearly $x^n = (-1)^n (-x)^n$, so by replacing x with $-x$ we get the thesis.

Basic Stirling number identities (5)

Reversing the formulas for falling and rising factorials

For every $n \geqslant 0$,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} \text{ and } x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

Corollary

$$\sum_k \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^{n-k} = \sum_k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^{n-k} = [m = n]$$

Indeed,

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} \left(\sum_m \left[{k \atop m} \right] x^m \right) = \sum_m \left(\sum_k \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^{n-k} \right) x^m$$

must hold for every x; the other equality is proved similarly.

Stirling's inversion formula (cf. Exercise 6.12)

Statement

Let f and g be two functions defined on $\mathbb{N}$ with values in $\mathbb{C}$.
The following are equivalent:

1. For every $n \geqslant 0$,

$$g(n) = \sum_k \left\{ {n \atop k} \right\} (-1)^k f(k).$$

2. For every $n \geqslant 0$,

$$f(n) = \sum_k \left[{n \atop k} \right] (-1)^k g(k).$$

Stirling's inversion formula (cf. Exercise 6.12)

Proof

If $g(n) = \sum_k \left\{ n \atop k \right\} (-1)^k f(k)$ for every $n \geqslant 0$, then also for $n \geqslant 0$

$$\begin{aligned}
\sum_k \left[n \atop k \right] (-1)^k g(k) &= \sum_k \left[n \atop k \right] (-1)^k \sum_m \left\{ k \atop m \right\} (-1)^m f(m) \\
&= \sum_{k,m} (-1)^{k+m} f(m) \left[n \atop k \right] \left\{ k \atop m \right\} \\
&= \sum_{k,m} (-1)^{2n-k-m} f(m) \left[n \atop k \right] \left\{ k \atop m \right\} \\
&= \sum_m (-1)^{n-m} f(m) \sum_k (-1)^{n-k} \left[n \atop k \right] \left\{ k \atop m \right\} \\
&= \sum_m (-1)^{n-m} f(m) [m = n] \\
&= f(n) .
\end{aligned}$$

The other implication is proved similarly.

Next subsection

Stirling's triangles in tandem

Basic recurrences of Stirling numbers yield for every integers k, n a simple law:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} -k \\ -n \end{Bmatrix} \qquad \text{with } \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = [n=0] \text{ and } \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{Bmatrix} 0 \\ k \end{Bmatrix} = [k=0]$$

n	$\begin{Bmatrix} n \\ -5 \end{Bmatrix}$	$\begin{Bmatrix} n \\ -4 \end{Bmatrix}$	$\begin{Bmatrix} n \\ -3 \end{Bmatrix}$	$\begin{Bmatrix} n \\ -2 \end{Bmatrix}$	$\begin{Bmatrix} n \\ -1 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 0 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 2 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 3 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 4 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 5 \end{Bmatrix}$
-5	1										
-4	10	1									
-3	35	6	1								
-2	50	11	3	1							
-1	24	6	2	1	1						
0	0	0	0	0	0	1					
1	0	0	0	0	0	0	1				
2	0	0	0	0	0	0	1	1			
3	0	0	0	0	0	0	1	3	1		
4	0	0	0	0	0	0	1	7	6	1	
5	0	0	0	0	0	0	1	15	25	10	1

Stirling numbers cheat sheet

- $\left\{ {n \atop 0} \right\} = \left[{n \atop 0} \right] = [n = 0]$
- $\left\{ {n \atop 1} \right\} = [n > 0]$ and $\left[{n \atop 1} \right] = (n-1)![n > 0]$
- $\left\{ {n \atop 2} \right\} = (2^{n-1} - 1)[n > 0]$ and $\left[{n \atop 2} \right] = (n-1)! H_{n-1} [n > 0]$
- $\left\{ {n \atop n-1} \right\} = \left[{n \atop n-1} \right] = \binom{n}{2} = \dfrac{n(n-1)}{2}$
- $\left\{ {n \atop n} \right\} = \left[{n \atop n} \right] = \binom{n}{n} = 1$
- $\left\{ {n \atop k} \right\} = \left[{n \atop k} \right] = \binom{n}{k} = 0$, if $k > n$ or $k < 0$
- $\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$ and $\left[{n \atop k} \right] = (n-1) \left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right]$
- $\sum_k \left\{ {n \atop k} \right\} x^{\underline{k}} = x^n$ and $\sum_k \left[{n \atop k} \right] x^k = x^{\overline{n}}$
- $\sum_k \left[{n \atop k} \right] = n!$
- $\sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}} = x^n$ and $\sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k = x^{\underline{n}}$
- $\sum_k \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^k = \sum_k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^k = [m = n]$

Next section

Fibonacci numbers: Idea

Fibonacci's problem

A pair of baby rabbits is left on an island.

- A baby rabbit becomes adult in one month.
- A pair of adult rabbits produces a pair of baby rabbits each month.

How many pairs of rabbits will be on the island ofter n months?
How many of them will be adult, and how many will be babies?

Leonardo Fibonacci (1175–1235)

Fibonacci numbers: Idea

Fibonacci's problem

A pair of baby rabbits is left on an island.

- A baby rabbit becomes adult in one month.
- A pair of adult rabbits produces a pair of baby rabbits each month.

How many pairs of rabbits will be on the island ofter n months?
How many of them will be adult, and how many will be babies?

Solution (see Exercise 6.6)

- On the first month, the two baby rabbits will have become adults.
- On the second month, the two adult rabbits will have produced a pair of baby rabbits.
- On the third month, the two adult rabbits will have produced *another* pair of baby rabbits, while the other two baby rabbits will have become adults.
- And so on, and so on . . .

Leonardo Fibonacci (1175–1235)

Fibonacci Numbers: Definition

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

Formulae for computing:

- $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$
- $f_n = \frac{\Phi^n - \hat{\Phi}^n}{\sqrt{5}}$ ("Binet form")

The golden ratio

The constant $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ is called golden ratio :

If a line segment a is divided into two sub-segments b and $a-b$ so that $a : b = b : (a-b)$, then

$$\frac{a}{b} = \Phi \text{ and } \frac{b}{a} = -\hat{\Phi}$$

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Generating Function for Fibonacci Numbers

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Applying Addition to some known generating functions:

$$\begin{array}{rlllllcc} & \langle 0, & 1, & 0, & 0, & 0, & \cdots \rangle & \leftrightarrow & x \\ & \langle 0, & f_0, & f_1, & f_2, & f_3, & \cdots \rangle & \leftrightarrow & xF(x) \\ + & \langle 0, & 0, & f_0, & f_1, & f_2, & \cdots \rangle & \leftrightarrow & x^2 F(x) \\ \hline & \langle 0, & 1 + f_0, & f_1 + f_0,, & f_2 + f_1, & f_3 + f_2, & \cdots \rangle & \leftrightarrow & x + xF(x) + x^2 F(x) \end{array}$$

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Closed form of the generating function: $F(x) = \frac{x}{1-x-x^2}$

Evaluation of Coefficients: Factorization

- We know from the previous lecture that

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

- Let's try to represent a generating function in the form:

$$\begin{aligned} G(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A \sum_{n \geqslant 0} (\alpha x)^n + B \sum_{n \geqslant 0} (\beta x)^n \\ &= \sum_{n \geqslant 0} (A\alpha^n + B\beta^n) x^n \end{aligned}$$

- The task is to find such constants A, B, α, β that

$$G(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{A - A\beta x + B - B\alpha x}{(1-\alpha x)(1-\beta x)}$$

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Factorization for Fibonacci (2)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\alpha x)(1-\beta x) & = & 1-x-x^2 \\ (A+B)-(A\beta+B\alpha)x & = & x \end{cases}$$

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To factorize $1-x-x^2$

- Solve the equation $w^2-wx-x^2=0$ (i.e. $w=1$ gives the special case $1-x-x^2=0$):

$$w_{1,2} = \frac{x \pm \sqrt{x^2+4x^2}}{2} = \frac{1 \pm \sqrt{5}}{2}x$$

- Therefore

$$w^2-wx-x^2 = \left(w - \frac{1+\sqrt{5}}{2}x\right)\left(w - \frac{1-\sqrt{5}}{2}x\right)$$

and

$$1-x-x^2 = \left(1 - \frac{1+\sqrt{5}}{2}x\right)\left(1 - \frac{1-\sqrt{5}}{2}x\right)$$

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A general trick

Let $p(x)=\sum_{k=0}^{n} a_k x^k$ be a polynomial over $\mathbb{C}$ of degree n such that $a_0 = p(0) \neq 0$.

- Then all the roots of p have a multiplicative inverse.
- Consider the "reverse" polynomial

$$p_R(x) = \sum_{k=0}^{n} a_k x^{n-k} = x^n p\left(\frac{1}{x}\right)$$

- Then α is a root of p if and only if $1/\alpha$ is a root of p_R, because if $p(x) = a_n(x-\alpha_1)\cdots(x-\alpha_n)$, then $p_R(x) = a_n(1-\alpha_1 x)\cdots(1-\alpha_n x)$.

Factorization for Fibonacci (3)

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Denote $\Phi = \frac{1+\sqrt{5}}{2}$ (golden ratio):

- "phi hat" is
$$\widehat{\Phi} = 1-\Phi = 1-\frac{1+\sqrt{5}}{2} = \frac{2-1-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$$
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Factorization for Fibonacci (4)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\Phi x)(1-\widehat{\Phi}x) & = & 1-x-x^2 \\ (A+B)-(A\widehat{\Phi}+B\Phi)x & = & x \end{cases}$$

To find A and B:

- Solve

$$\begin{cases} A+B=0 \\ A\widehat{\Phi}+B\Phi=-1 \end{cases}$$

- This is $A=1/(\Phi-\widehat{\Phi})$:

$$\begin{aligned} A &= 1/(\Phi-\widehat{\Phi}) \\ &= 1/\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right) \\ &= \frac{2}{1+\sqrt{5}-1+\sqrt{5}} = \frac{1}{\sqrt{5}} \end{aligned}$$

Factorization for Fibonacci (4)

- For the generating function of Fibonacci Numbers we need to solve the equations:

$$\begin{cases} (1-\Phi x)(1-\widehat{\Phi}x) & = & 1-x-x^2 \\ (A+B)-(A\widehat{\Phi}+B\Phi)x & = & x \end{cases}$$

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Factorization for Fibonacci (5)

To conclude:

- We have $\alpha = \Phi = (1+\sqrt{5})/2$, $\beta = \widehat{\Phi} = (1-\sqrt{5})/2$, $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$
- Generating function

$$G(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\Phi x} - \frac{1}{1-\widehat{\Phi} x}\right)$$

- Closed formula for f_n

$$f_n = A\alpha^n + B\beta^n = \frac{1}{\sqrt{5}}\left(\Phi^n - \widehat{\Phi}^n\right)$$