Sums

ITT9131 Konkreetne Matemaatika

Contents

Next section

Multiple sums

Definition

$$\sum_{i,j} a_{ij} = \sum_i \left(\sum_j a_{ij} \, [P(i,j)] \right)$$

where P is a predicate $P(i,j) = (i \in K_1) \wedge (j \in K_2)$ for sets for indexes K_1 and K_2.

Multiple sums

Definition

$$\sum_{i,j} a_{ij} = \sum_i \left(\sum_j a_{ij} \, [P(i,j)] \right)$$

where P is a predicate $P(i,j) = (i \in K_1) \wedge (j \in K_2)$ for sets for indexes K_1 and K_2.

Generalisation of the law of associativity (law of *interchanging the order of summation*):

$$\sum_j \sum_k a_{j,k} \, [P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} \, [P(j,k)]$$

Multiple sums

Definition

$$\sum_{i,j} a_{ij} = \sum_{i} \left(\sum_{j} a_{ij} \, [P(i,j)] \right)$$

where P is a predicate $P(i,j) = (i \in K_1) \wedge (j \in K_2)$ for sets for indexes K_1 and K_2.

If $a_{jk} = a_j a_k$, then

- $$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} b_k \right)$$

Multiple sums with independent indices

If $P(j,k) = Q(j) \wedge R(k)$, where Q and R are properties and $\wedge$ indicates the logical conjunction (AND), then the indices j and k are independent and the double sum can be rewritten:

$$
\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k} \left([Q(j) \wedge R(k)] \right) \\
&= \sum_{j,k} a_{j,k} [Q(j)][R(k)] \\
&= \sum_{j} [Q(j)] \sum_{k} a_{j,k} R(k) = \sum_{j} \sum_{k} a_{j,k} \\
&= \sum_{k} a_{j,k} [R(k)] \sum_{j} [Q(j)] = \sum_{k} \sum_{j} a_{j,k}
\end{aligned}
$$

Multiple sums with dependent indices

In general, the indices are not independent, but we can write:

$$P(j,k) = Q(j) \wedge R'(j,k) = R(k) \wedge Q'(j,k)$$

In this case, we can proceed as follows:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}[Q(j)][R'(j,k)] \\
&= \sum_{j}[Q(j)]\sum_{k} a_{j,k}[R'(j,k)] = \sum_{j\in J}\sum_{k\in K'} a_{j,k} \\
&= \sum_{k}[R(k)]\sum_{j} a_{j,k}[Q'(j,k)] = \sum_{k\in K}\sum_{j\in J'} a_{j,k}
\end{aligned}$$

where:

- $J = \{j \mid Q(j)\}, K' = \{k \mid R'(j,k)\}$
- $K = \{k \mid R(k)\}, J' = \{j \mid Q'(j,k)\}$

Warmup: what's wrong with this sum?

$$\begin{aligned}
\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) &= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} 1 \\
&= n^2
\end{aligned}$$

Warmup: what's wrong with this sum?

$$
\begin{aligned}
\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) &= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} 1 \\
&= n^2
\end{aligned}
$$

Solution

The second passage is seriously wrong:
It is not licit to turn two independent variables into two dependent ones.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \le j \le n} \sum_{j \le k \le n} a_j a_k$.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n}\sum_{k=j}^{n} a_j a_k = \sum_{1\le j\le n}\sum_{j\le k\le n} a_j a_k$.

A crucial observation

$$[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]$$

Hence,

$$\sum_{j=1}^{n}\sum_{k=j}^{n} a_j a_k = \sum_{k=1}^{n}\sum_{j=1}^{k} a_j a_k$$

Also,

$$[1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le j, k \le n] + [1 \le j = k \le n]$$

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \le j \le n} \sum_{j \le k \le n} a_j a_k$.

A crucial observation (cont.)

This can also be understood by considering the following matrix:

$$\begin{pmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 & \ldots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & a_2 a_3 & \ldots & a_2 a_n \\ a_3 a_1 & a_3 a_2 & a_3 a_3 & \ldots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \ldots & a_n a_n \end{pmatrix}$$

and observing that $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = S_U$ is the sum of the elements of the upper triangular part of the matrix.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n}\sum_{k=j}^{n} a_j a_k = \sum_{1\le j\le n}\sum_{j\le k\le n} a_j a_k$.

A crucial observation (end)

If we add to S_U the sum $S_L = \sum_{k=1}^{n}\sum_{j=1}^{k} a_j a_k$ of the elements of the lower triangular part of the matrix, we count each element of the matrix once, except those on the main diagonal, which we count twice.
But the matrix is symmetric, so $S_U = S_L$, and

$$S_U = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right)$$

Examples of multiple summing

Example 1

$$
\begin{aligned}
S_n &= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j < k} \frac{1}{k-j} \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant k-j < k} \frac{1}{j} \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{0 < j \leqslant k-1} \frac{1}{j} \\
&= \sum_{1 \leqslant k \leqslant n} H_{k-1} \\
&= \sum_{1 \leqslant k+1 \leqslant n} H_k \\
&= \sum_{0 \leqslant k < n} H_k
\end{aligned}
$$

Examples of multiple summing

Example 2

$$
\begin{aligned}
S_n &= \sum_{1 \leqslant j \leqslant n} \sum_{j < k \leqslant n} \frac{1}{k-j} \\
&= \sum_{1 \leqslant j \leqslant n} \sum_{j < k+j \leqslant n} \frac{1}{k} \\
&= \sum_{1 \leqslant j \leqslant n} \sum_{0 < k \leqslant n-j} \frac{1}{k} \\
&= \sum_{1 \leqslant j \leqslant n} H_{n-j} \\
&= \sum_{1 \leqslant n-j \leqslant n} H_j \\
&= \sum_{0 \leqslant j < n} H_j
\end{aligned}
$$

Examples of multiple summing

Example 3

$$
\begin{aligned}
S_n &= \sum_{1\leqslant j<k\leqslant n} \frac{1}{k-j} \\
&= \sum_{1\leqslant j<k+j\leqslant n} \frac{1}{k} \\
&= \sum_{1\leqslant k\leqslant n} \; \sum_{1\leqslant j\leqslant n-k} \frac{1}{k} \\
&= \sum_{1\leqslant k\leqslant n} \frac{n-k}{k} \\
&= \sum_{1\leqslant k\leqslant n} \frac{n}{k} - \sum_{1\leqslant k\leqslant n} 1 \\
&= n\left(\sum_{1\leqslant k\leqslant n} \frac{1}{k}\right) - n = nH_n - n
\end{aligned}
$$

We have proved: $\sum_{0\leqslant k<n} H_k = nH_n - n$

Examples of multiple summing

Example 3

$$
\begin{aligned}
S_n &= \sum_{1 \leqslant j < k \leqslant n} \frac{1}{k-j} \\
&= \sum_{1 \leqslant j < k+j \leqslant n} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j \leqslant n-k} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n-k}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n}{k} - \sum_{1 \leqslant k \leqslant n} 1 \\
&= n \left(\sum_{1 \leqslant k \leqslant n} \frac{1}{k} \right) - n = nH_n - n
\end{aligned}
$$

We have proved: $\sum_{0 \leqslant k < n} H_k = nH_n - n$

Next section

General Methods: a Review

Example

$$\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
$\square_n$	0	1	5	14	30	55	91	140	204	285	385	506	650

Next subsection

Review: Method 0

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (`http://oeis.org/`)
- *etc*

Possible answer:

$$\square_n = \frac{n(n+1)(2n+1)}{6} \qquad \text{for } n \geqslant 0$$

Next subsection

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \quad$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n / n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n / n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90
$3\Box_n / n(n+1)$	–	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

Review: Method 1

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\square_n$	0	1	5	14	30	55	91	140	204	285
$\square_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\square_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90
$3\square_n/n(n+1)$	–	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

Hypothesis:

$$\frac{3\square_n}{n(n+1)} = n + \frac{1}{2} \qquad \Longrightarrow$$

$$\square_n = \frac{n(n+1/2)(n+1}{3} = \frac{n(n+1)(2n+1)}{6}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$\begin{aligned} 3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ &= n(n+\frac{1}{2})(n+1) \end{aligned}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$\begin{aligned} 3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ &= n(n+\frac{1}{2})(n+1) \end{aligned}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$\begin{aligned} 3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ &= n(n+\frac{1}{2})(n+1) \end{aligned}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$\begin{aligned}
3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\
&= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\
&= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
&= n(n+\frac{1}{2})(n+1)
\end{aligned}$$

Q.E.D.

Next subsection

Review: Method 2

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\boxdot_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\boxdot_n + (n+1)^3 = \sum_{0 \leqslant k \leqslant n} (k+1)^3 = \sum_{0 \leqslant k \leqslant n} (k^3 + 3k^2 + 3k + 1)$$

$$= \boxdot_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Delete $\boxdot_n$ and extract $\square_n$

$$3\square_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$

$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$

Review: Method 2

Example; $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\boxtimes_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\boxtimes_n + (n+1)^3 = \sum_{0\leqslant k\leqslant n} (k+1)^3 = \sum_{0\leqslant k\leqslant n} (k^3 + 3k^2 + 3k + 1)$$
$$= \boxtimes_n + 3\Box_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Delete $\boxtimes_n$ and extract $\Box_n$

$$3\Box_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$
$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$

Review: Method 2

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\Cube_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\begin{aligned}\Cube_n + (n+1)^3 &= \sum_{0 \leqslant k \leqslant n} (k+1)^3 = \sum_{0 \leqslant k \leqslant n} (k^3 + 3k^2 + 3k + 1)\\ &= \Cube_n + 3\Box_n + 3\frac{(n+1)n}{2} + (n+1).\end{aligned}$$

- Delete $\Cube_n$ and extract $\Box_n$

$$\begin{aligned}3\Box_n &= (n+1)^3 - 3(n+1)n/2 - (n+1)\\ &= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)\end{aligned}$$

Review: Method 2

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\cube_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\cube_n + (n+1)^3 = \sum_{0 \leqslant k \leqslant n} (k+1)^3 = \sum_{0 \leqslant k \leqslant n} (k^3 + 3k^2 + 3k + 1)$$

$$= \cube_n + 3\Box_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Delete $\cube_n$ and extract $\Box_n$

$$3\Box_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$

$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$

Next subsection

Review: Method 3

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Review: Method 3

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n$

- Equation: $n = n - 1 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = 1; \beta = \gamma = 0$,
- and the solution has a form $n = A(n)$.

Review: Method 3

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n^2$

- Equation: $n^2 = (n-1)^2 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = -1; \beta = 2; \gamma = 0$,
- and hence, the solution has a form $n^2 = -A(n) + 2B(n) = -n + 2B(n)$.
- That gives $B(n) = \frac{n^2+n}{2}$.

Review: Method 3

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n^3$

- Equation: $n^3 = (n-1)^3 + \alpha + \beta n + \gamma n^2 = n^3 - 3n^2 + 3n - 1 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = 1; \beta = -3; \gamma = 3$,
- hence, $n^3 = A(n) - 3B(n) + 3C(n) = n - 3\frac{n^2+n}{2} + 3C(n)$.
- That gives $6C(n) = 2n^3 - 2n + 3n^2 + 3n = 2n^3 + 3n^2 + n = n(2n+1)(n+1)$.

Review: Method 3

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \quad$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$
$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

To resume

- $R_n = \Box_n$ iff $\alpha = \beta = 0; \gamma = 1$
- The solution is

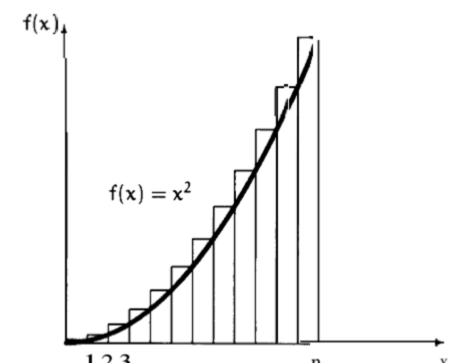
$$\Box_n = \frac{n(n+1)(2n+1)}{6}$$

Next subsection

Review: Method 4

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.



$$\int_0^n x^2 \mathrm{d}x = \frac{n^3}{3} \tag{1}$$

$$\square_n = \int_0^n x^2 \mathrm{d}x + E_n \tag{2}$$

$$E_n = \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \mathrm{d}x \right) \tag{3}$$

Review: Method 4

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.

Evaluate (3):

$$E_n = \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \mathrm{d}x \right)$$
$$= \sum_{k=1}^{n} \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right)$$
$$= \sum_{k=1}^{n} \left(k - \frac{1}{3} \right)$$
$$= \frac{(n+1)n}{2} - \frac{n}{3} = \frac{3n^2 + n}{6}.$$

Finally, from (2) and (1) we get :

$$\square_n = \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Next subsection

Review: Method 5

Example; $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Expand and Contract.

$$\begin{aligned}
\Box_n = \sum_{0\leqslant k\leqslant n} k^2 &= \sum_{0\leqslant j\leqslant k\leqslant n} k \\
&= \sum_{0\leqslant j\leqslant n} \sum_{j\leqslant k\leqslant n} k \\
&= \sum_{0\leqslant j\leqslant n} \frac{(j+n)(n-j+1)}{2} \\
&= \frac{1}{2} \sum_{0\leqslant j\leqslant n} (j - j^2 + n(n+1)) \\
&= \frac{1}{4} n(n+1) - \frac{1}{2}\Box_n + \frac{1}{2} n^2(n+1)
\end{aligned}$$

Review: Method 5

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Expand and Contract.

$$\begin{aligned}
\Box_n &= \sum_{0 \leqslant k \leqslant n} k^2 = \sum_{0 \leqslant j \leqslant k \leqslant n} k \\
&= \sum_{0 \leqslant j \leqslant n} \sum_{j \leqslant k \leqslant n} k \\
&= \sum_{0 \leqslant j \leqslant n} \frac{(j+n)(n-j+1)}{2} \\
&= \frac{1}{2} \sum_{0 \leqslant j \leqslant n} \left(j - j^2 + n(n+1)\right) \\
&= \frac{1}{4} n(n+1) - \frac{1}{2} \Box_n + \frac{1}{2} n^2 (n+1)
\end{aligned}$$

Next subsection

Review: Other methods

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

- Finite calculus
- Generating functions

Next section

Next subsection

Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibnitz's notation If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibnitz's notation If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

In general, if $h \in \mathbb{R}$ (or $h \in \mathbb{C}$), then

Forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

Backward difference

$$\nabla_h[f](x) = f(x) - f(x-h)$$

Central difference

$$\delta_h[f](x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h)$$

Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

In general, if $h \in \mathbb{R}$ (or $h \in \mathbb{C}$), then

Forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

Backward difference

$$\nabla_h[f](x) = f(x) - f(x-h)$$

Central difference

$$\delta_h[f](x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h)$$

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{\Delta_h[f](x)}{h}$$

Derivative of Power function

Example: $f(x) = x^3$

In this case

$$\begin{aligned}\Delta_h[f](x) &= f(x+h) - f(x)\\ &= (x+h)^3 - x^3\\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3\\ &= h(3x^2 + 3xh + h^2)\end{aligned}$$

Hence

$$\mathrm{D}f(x) = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Derivative of Power function

Example: $f(x) = x^3$

In this case

$$\begin{aligned}\Delta_h[f](x) &= f(x+h) - f(x)\\ &= (x+h)^3 - x^3\\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3\\ &= h(3x^2 + 3xh + h^2)\end{aligned}$$

Hence

$$\mathrm{D}f(x) = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

In general:

$$\mathrm{D}(x^m) = mx^{m-1}$$

(Forward) Difference of Power Function

Example: $f(x) = x^3$

In this case

$$\Delta f(x) = \Delta_1[f](x) = 3x^2 + 3x + 1$$

(Forward) Difference of Power Function

Example: $f(x) = x^3$

In this case

$$\Delta f(x) = \Delta_1 [f](x) = 3x^2 + 3x + 1$$

In general:

$$\Delta(x^m) = \sum_{k=1}^{m} \binom{m}{k} x^{m-k}$$

Falling and Rising Factorials

Definition

The falling factorial power (or simply falling factorial) is defined for $m \geqslant 0$ by

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

The rising factorial power (or simply rising factorial) is defined for $m \geqslant 0$ by

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

Properties

$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}}$$

$$n! = n^{\underline{n}}$$

$$n! = 1^{\overline{n}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$x^{\underline{m}} \cdot x^{\underline{n}} = x^{\underline{n}} \cdot x^{\underline{m}} = (x^{\underline{m}})^2 (x-m)^{\underline{n-m}}, \text{ for } n > m$$

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

$$x^{\underline{m}} = \frac{x^{\underline{m+1}}}{x-m}$$

$$x^{\underline{-m}} = \frac{x}{x^{\overline{m+1}}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$

Falling and Rising Factorials

Definition

The falling factorial power (or simply falling factorial) is defined for $m \geqslant 0$ by

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

The rising factorial power (or simply rising factorial) is defined for $m \geqslant 0$ by

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

Properties

$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}}$$

$$n! = n^{\underline{n}}$$

$$n! = 1^{\overline{n}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$x^{\underline{m}} \cdot x^{\underline{n}} = x^{\underline{n}} \cdot x^{\underline{m}} = (x^{\underline{m}})^2(x-m)^{\underline{n-m}}, \text{ for } n > m$$

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

$$x^{\underline{m}} = \frac{x^{\underline{m+1}}}{x-m}$$

$$x^{\underline{-m}} = \frac{x}{x^{\underline{m+1}}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$

Warmup: what is $0^{\underline{m}}$?

Case 1: $m > 0$

Then $0^{\underline{m}} = 0 \cdot (-1)^{\underline{m-1}} = 0$.

Case 2: $m = 0$

Then $0^{\underline{m}} = 1$ because it is defined as an empty product.

Case 3: $m < 0$

Then we want $1 = 0^{\underline{0}} = 0^{\underline{m}} \cdot (0-m)^{\underline{-m}}$.
But as $m < 0$, $(-m)^{\underline{-m}} = |m|!$, and

$$0^{\underline{m}} = 1/|m|!$$

Warmup: what is $0^{\underline{m}}$?

Case 1: $m > 0$

Then $0^{\underline{m}} = 0 \cdot (-1)^{\underline{m-1}} = 0$.

Case 2: $m = 0$

Then $0^{\underline{m}} = 1$ because it is defined as an empty product.

Case 3: $m < 0$

Then we want $1 = 0^{\underline{0}} = 0^{\underline{m}} \cdot (0-m)^{\underline{-m}}$.
But as $m < 0$, $(-m)^{\underline{-m}} = |m|!$, and

$$0^{\underline{m}} = 1/|m|!$$

Warmup: what is $0^{\underline{m}}$?

Case 1: $m > 0$

Then $0^{\underline{m}} = 0 \cdot (-1)^{\underline{m-1}} = 0$.

Case 2: $m = 0$

Then $0^{\underline{m}} = 1$ because it is defined as an empty product.

Case 3: $m < 0$

Then we want $1 = 0^{\underline{0}} = 0^{\underline{m}} \cdot (0 - m)^{\underline{-m}}$.
But as $m < 0$, $(-m)^{\underline{-m}} = |m|!$, and

$$0^{\underline{m}} = 1/|m|!$$

Difference of Falling Power Function

$$
\begin{aligned}
\Delta(x^{\underline{m}}) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\
&= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+2)(x-m+1) \\
&= x(x-1)\cdots(x-m+2)((x+1)-(x-m+1)) \\
&= mx(x-1)\cdots(x-m+2) \\
&= mx^{\underline{m-1}}
\end{aligned}
$$

Difference of Falling Power Function

$$\begin{aligned}
\Delta(x^{\underline{m}}) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\
&= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+2)(x-m+1) \\
&= x(x-1)\cdots(x-m+2)((x+1)-(x-m+1)) \\
&= mx(x-1)\cdots(x-m+2) \\
&= mx^{\underline{m-1}}
\end{aligned}$$

Hence

$$\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}$$

Difference of Falling Power Function (2)

Let's check this formula for negative power:

$$\begin{aligned}
\Delta x^{\underline{-2}} &= (x+1)^{\underline{-2}} - x^{\underline{-2}} \\
&= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\
&= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)} \\
&= -\frac{2}{(x+1)(x+2)(x+3)} \\
&= -2x^{\underline{-3}}
\end{aligned}$$

Next subsection

Indefinite Integrals and Sums

Fundamental Theorem of Calculus

$$\mathrm{D}f(x) = g(x) \qquad \text{iff} \qquad \int g(x)\mathrm{d}x = f(x) + \mathrm{C}$$

Definition

The indefinite sum of function $g(x)$ is a class of functions those difference is $g(x)$:

$$\Delta f(x) = g(x) \qquad \text{iff} \qquad \sum g(x)\delta x = f(x) + C(x),$$

where $C(x)$ is a "periodic function" such that $C(x+1) = C(x)$ for any integer value of x.

Indefinite Integrals and Sums

Fundamental Theorem of Calculus

$$\mathrm{D}f(x) = g(x) \quad \text{iff} \quad \int g(x)\mathrm{d}x = f(x) + \mathrm{C}$$

Definition

The indefinite sum of function $g(x)$ is a class of functions those difference is $g(x)$:

$$\Delta f(x) = g(x) \quad \text{iff} \quad \sum g(x)\delta x = f(x) + C(x),$$

where $C(x)$ is a "periodic function" such that $C(x+1) = C(x)$ for any integer value of x.

Definite Integrals and Sums

If $g(x) = \mathrm{D}f(x)$, then

$$\int_a^b g(x)\mathrm{d}x = f(x)\Big|_a^b = f(b) - f(a)$$

Analogously:

If $g(x) = \Delta f(x)$, then

$$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a)$$

Definite Integrals and Sums

If $g(x) = \mathrm{D}f(x)$, then

$$\int_a^b g(x)\mathrm{d}x = f(x)\Big|_a^b = f(b) - f(a)$$

Analogously:

If $g(x) = \Delta f(x)$, then

$$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a)$$

Definite sums

Observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x =$
  $(f(b+1) - f(a)) - (f(b) - f(a)) = f(b+1) - f(b) = g(b)$

Definite sums

Observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x =$
  $(f(b+1) - f(a)) - (f(b) - f(a)) = f(b+1) - f(b) = g(b)$

Definite sums

Observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x =$
  $(f(b+1) - f(a)) - (f(b) - f(a)) = f(b+1) - f(b) = g(b)$

Definite sums

Observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x =$ $(f(b+1) - f(a)) - (f(b) - f(a)) = f(b+1) - f(b) = g(b)$

Hence

$$\begin{aligned}
\sum_a^b g(x)\delta x &= \sum_{k=a}^{b-1} g(k) = \sum_{a \leqslant k < b} g(k) \\
&= \sum_{a \leqslant k < b} (f(k+1) - f(k)) = \\
&= (f(a+1) - f(a)) + (f(a+2) - f(a+1)) + \cdots \\
&\qquad + (f(b-1) - f(b-2)) + (f(b) - f(b-1)) \\
&= f(b) - f(a)
\end{aligned}$$

Integrals and Sums of Powers

If $m \neq -1$, then

$$\int_0^n x^m \mathrm{d}x = \frac{x^{m+1}}{m+1}\Big|_0^n = \frac{n^{m+1}}{m+1}$$

Analogous finite case:

If $m \neq -1$, then

$$\sum_0^n k^m \delta x = \sum_{0 \leqslant k < n} k^m = \frac{k^{m+1}}{m+1}\Big|_0^n = \frac{n^{m+1}}{m+1}$$

Integrals and Sums of Powers

If $m \neq -1$, then

$$\int_0^n x^m \mathrm{d}x = \frac{x^{m+1}}{m+1}\bigg|_0^n = \frac{n^{m+1}}{m+1}$$

Analogous finite case:

If $m \neq -1$, then

$$\sum_0^n k^{\underline{m}} \delta x = \sum_{0 \leqslant k < n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\bigg|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

Sums of Powers: applications

Case $m = 1$

$$\sum_{0 \leqslant k < n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

Case $m = 2$ Due to $k^2 = k^{\underline{2}} + k^{\underline{1}}$ we get

$$\begin{aligned}
\sum_{0 \leqslant k < n} k^2 &= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\
&= \frac{1}{3} n(n-1)(n-2) + \frac{1}{2} n(n-1) \\
&= \frac{1}{6} n(2(n-1)(n-2) + 3(n-1)) \\
&= \frac{1}{6} n(n-1)(2n-4+3) \\
&= \frac{1}{6} n(n-1)(2n-1)
\end{aligned}$$

Taking $n+1$ instead of n gives

$$\Box_n = \frac{(n+1)n(2n+1)}{6}$$

Sums of Powers: applications

Case $m = 1$

$$\sum_{0 \leqslant k < n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

Case $m = 2$ Due to $k^2 = k^{\underline{2}} + k^{\underline{1}}$ we get

$$\begin{aligned}\sum_{0 \leqslant k < n} k^2 &= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\ &= \frac{1}{3} n(n-1)(n-2) + \frac{1}{2} n(n-1) \\ &= \frac{1}{6} n(2(n-1)(n-2) + 3(n-1)) \\ &= \frac{1}{6} n(n-1)(2n-4+3) \\ &= \frac{1}{6} n(n-1)(2n-1)\end{aligned}$$

Taking $n+1$ instead of n gives

$$\Box_n = \frac{(n+1)n(2n+1)}{6}$$

Sums of Powers (case $m = -1$)

Note

$$\begin{aligned}
\Delta H_x &= H_{x+1} - H_x \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x} + \frac{1}{x+1} - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{x} \\
&= \frac{1}{x+1}
\end{aligned}$$

and

$$\sum_a^b x^{\underline{-1}} \delta x = H_x \Big|_a^b$$

Sums of Discrete Exponential Functions

- We have
$$\mathrm{D}e^x = e^x.$$
Finite analogue should have $\Delta f(x) = f(x)$. That means:
$$f(x+1) - f(x) = f(x) \qquad \text{iff} \qquad f(x+1) = 2f(x) \qquad \text{iff} \qquad f(x) = 2^x$$
- The difference of c^x is
$$\Delta(c^x) = c^{x+1} - c^x = (c-1)c^x$$
and anti-difference is then $c^x/(c-1)$, if $c \neq 1$ that gives the the sum of geometric progression
$$\sum_{a \leqslant k < b} c^k = \sum_a^b c^x \delta x = \frac{c^x}{c-1}\Big|_a^b = \frac{c^b - c^a}{c-1}$$

Next subsection

Summation by Parts

Infinite analogue: integration by parts:

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

Difference of product:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)\\ &= u(x)\Delta v(x) + v(x+1)\Delta u(x)\\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x)\end{aligned}$$

where E is the shift operator $Ef(x) = f(x+1)$. Taking the indefinite sum from both sides yields

Rule for summation by parts:

$$\sum u\Delta v = uv - \sum Ev\Delta u$$

Summation by Parts

Infinite analogue: integration by parts:

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

Difference of product:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)\\ &= u(x)\Delta v(x) + v(x+1)\Delta u(x)\\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x)\end{aligned}$$

where E is the shift operator $Ef(x) = f(x+1)$. Taking the indefinite sum from both sides yields

Rule for summation by parts:

$$\sum u\Delta v = uv - \sum Ev\Delta u$$

Warmup: why the asymmetry?

How is it that, in the rule for summation by parts:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

the left-hand side is symmetric in u and v, but the right-hand side is not?

Warmup: why the asymmetry?

How is it that, in the rule for summation by parts:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

the left-hand side is symmetric in u and v, but the right-hand side is not?

Because the symmetry is elsewhere!

We can also write:

$$\begin{aligned}
\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x) \\
&= Eu(x)\Delta v(x) + v(x)\Delta u(x)
\end{aligned}$$

So there actually is a symmetry—just not the one we thought:

$$u\Delta v + Ev\Delta u = v\Delta u + Eu\Delta v$$

Summation by Parts (2)

Example: $S = \sum_{k=0}^{n} k2^k$

- Taking $u(x) = x, v(x) = 2^x$ and $Ev(x) = 2^{x+1}$:

$$\sum x2^x \delta x = x2^x - \sum 2^{x+1} \delta x = x2^x - 2^{x+1} + C$$

This yields

$$\begin{aligned}
\sum_{k=0}^{n} k2^k &= \sum_{0}^{n+1} x2^x \delta x \\
&= \left(x2^x - 2^{x+1}\right)\Big|_0^{n+1} \\
&= \left((n+1)2^{n+1} - 2^{n+2}\right) - \left(0 \cdot 2^0 - 2\right) \\
&= (n-1)2^{n+1} + 2
\end{aligned}$$

Summation by Parts (3)

Example: $S = \sum_{k=0}^{n-1} kH_k$

Continuous analogue:

$$\begin{aligned}
\int x \ln x \, \mathrm{d}x &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, \mathrm{d}x \\
&= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, \mathrm{d}x \\
&= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} \\
&= \frac{x^2}{2} (\ln x - \frac{1}{2})
\end{aligned}$$

Summation by Parts (3)

Example: $S = \sum_{k=0}^{n-1} kH_k$

- Taking $u(x) = H_x$, $v(x) = \frac{x^{\underline{2}}}{2}$, $\Delta u(x) = \Delta H_x = x^{\underline{-1}} = \frac{1}{x+1}$, $\Delta v(x) = x = x^{\underline{1}}$ and $Ev(x) = \frac{(x+1)^{\underline{2}}}{2}$, we get

$$\begin{aligned}
\sum_{k=0}^{n-1} kH_k &= \sum_0^n xH_x \,\delta x = uv\Big|_0^n + \sum_0^n Ev\Delta u \,\delta x \\
&= \frac{x^{\underline{2}}}{2} H_x\Big|_0^n - \sum_0^n \frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}} \,\delta x \\
&= \frac{x^{\underline{2}}}{2} H_x\Big|_0^n - \frac{1}{2}\sum_0^n x \,\delta x \\
&= \left(\frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \cdot \frac{x^{\underline{2}}}{2}\right)\Big|_0^n \\
&= \frac{n^{\underline{2}}}{2}(H_n - \frac{1}{2})
\end{aligned}$$

Next section

Instead of Conclusion: Table of Differences

$f = \Sigma g$	$\Delta f = g$	$f = \Sigma g$	$\Delta f = g$
$x^{\underline{0}} = 1$	0	2^x	2^x
$x^{\underline{1}} = x$	1	c^x	$(c-1)c^x$
$x^{\underline{2}} = x(x-1)$	$2x$	$c^x/(c-1)$	c^x
$x^{\underline{m}}$	$mx^{\underline{m-1}}$	cf	$c\Delta f$
$x^{\underline{m+1}}/(m+1)$	$x^{\underline{m}}$	$f+g$	$\Delta f + \Delta g$
H_x	$x^{\underline{-1}} = 1/(x+1)$	fg	$f\Delta g + Eg\Delta f$

Next section

How to sum infinite number sequences?

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

'

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2.$$

How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots .$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \cdots = T - 1,$$

and

$$T = -1.$$

How to sum infinite number sequences?

Example 3

Let

$$\sum_{k\geqslant 0}(-1)^k = 1-1+1-1+1-1+1-1+\cdots =?$$

Different ways to sum

$$(1-1)+(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+0+\cdots = 0$$

and

$$1-(1-1)-(1-1)-(1-1)-(1-1)-\cdots = 1-0-0-0-0-0-\cdots = 1$$

Defining Infinite Sums: Nonnegative values

Definition 1

If $a_k \geqslant 0$ for every $k \geqslant 0$, then

$$\sum_{k \geqslant 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k = \sup_{K \subseteq \mathbb{N} \text{ finite}} \sum_{k \in K} a_k$$

Commutative property for infinite sums of nonnegative values

If $a_k \geqslant 0$ for every $k \geqslant 0$ and $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation of the set of natural numbers, then

$$\sum_{k \geqslant 0} a_k = \sum_{k \geqslant 0} a_{\sigma(k)}$$

Defining Infinite Sums: Nonnegative values

Definition 1

If $a_k \geqslant 0$ for every $k \geqslant 0$, then

$$\sum_{k \geqslant 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k = \sup_{K \subseteq \mathbb{N} \text{ finite}} \sum_{k \in K} a_k$$

Commutative property for infinite sums of nonnegative values

If $a_k \geqslant 0$ for every $k \geqslant 0$ and $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation of the set of natural numbers, then

$$\sum_{k \geqslant 0} a_k = \sum_{k \geqslant 0} a_{\sigma(k)}$$

Defining Infinite Sums: Arbitrary values

Every real number x can be written in the form $x = x^+ - x^-$, where:

- $x^+ = x \cdot [x > 0] = \max(0, x)$ is the *positive part* of x.
- $x^- = -x \cdot [x < 0] = \max(0, -x)$ is the negative part of x.

Observe that $|x| = x^+ + x^-$.

Definition 2

Let K be a set and a_k a real number for every $k \in K$, then:

$$\sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-$$

Defining Infinite Sums: Arbitrary values

Every real number x can be written in the form $x = x^+ - x^-$, where:

- $x^+ = x \cdot [x > 0] = \max(0, x)$ is the *positive part* of x.
- $x^- = -x \cdot [x < 0] = \max(0, -x)$ is the negative part of x.

Observe that $|x| = x^+ + x^-$.

Definition 2

Let K be a set and a_k a real number for every $k \in K$, then:

$$\sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-$$

Convergent and divergent series

The sum $\sum_k a_k$ is:

- absolutely convergent if both $\sum a_k^+$ and $\sum a_k^-$ are finite.
  In this case, $\sum_{k\in\mathbb{N}} a_k^+ - \sum_{k\in\mathbb{N}} a_k^- = \lim_{n\to\infty} \sum_{k=0}^n a_k$, and the infinite sum has the commutative property.
- positively divergent if $\sum a_k^+ = \infty$ and $\sum a_k^-$ is finite.
  In this case, it is licit to put $\sum_k a_k = +\infty$.
- negatively divergent if $\sum a_k^- = \infty$ and $\sum a_k^+$ is finite.
  In this case, it is licit to put $\sum_k a_k = -\infty$.

If both $\sum a_k^+ = \infty$ and $\sum a_k^- = \infty$, then "all bets are off":

Riemann series theorem

Let $\{a_k\}_{k\geqslant 0}$ a sequence of real numbers such that:

$$\sum_{k\geqslant 0} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = S \in \mathbb{R} \text{ but } \sum_{k\geqslant 0} |a_k| = +\infty$$

For every $M \in \mathbb{R}$ there exists a permutation σ of $\mathbb{N}$ such that:

$$\sum_{k\geqslant 0} a_{\sigma(k)} = \lim_{n\to\infty} \sum_{k=0}^{n} a_{\sigma(k)} = M$$

Convergent and divergent series

The sum $\sum_k a_k$ is:

- absolutely convergent if both $\sum a_k^+$ and $\sum a_k^-$ are finite.
  In this case, $\sum_{k\in\mathbb{N}} a_k^+ - \sum_{k\in\mathbb{N}} a_k^- = \lim_{n\to\infty}\sum_{k=0}^n a_k$, and the infinite sum has the commutative property.
- positively divergent if $\sum a_k^+ = \infty$ and $\sum a_k^-$ is finite.
  In this case, it is licit to put $\sum_k a_k = +\infty$.
- negatively divergent if $\sum a_k^- = \infty$ and $\sum a_k^+$ is finite.
  In this case, it is licit to put $\sum_k a_k = -\infty$.

If both $\sum a_k^+ = \infty$ and $\sum a_k^- = \infty$, then "all bets are off":

Riemann series theorem

Let $\{a_k\}_{k\geqslant 0}$ a sequence of real numbers such that:

$$\sum_{k\geqslant 0} a_k = \lim_{n\to\infty}\sum_{k=0}^n a_k = S \in \mathbb{R} \text{ but } \sum_{k\geqslant 0} |a_k| = +\infty$$

For every $M \in \mathbb{R}$ there exists a permutation σ of $\mathbb{N}$ such that:

$$\sum_{k\geqslant 0} a_{\sigma(k)} = \lim_{n\to\infty}\sum_{k=0}^n a_{\sigma(k)} = M$$