Concrete Mathematics
Exercises from 20 September

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Exercise 1.2

Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg C, using the middle peg B as a help, if direct moves between A and B are disallowed.

Solution. For $n = 1$ the shortest sequence is $A \to B$, $B \to C$. For $n = 2$ it is:

1. $A \to B$.
2. $B \to C$.
3. $A \to B$.
4. $C \to B$. Note that the whole tower is on peg B now.
5. $B \to A$.
6. $B \to C$.
7. $A \to B$.
8. $B \to C$.

For the general case, observe that the strategy that solves the problem for n disks works as follows:

1. Move the upper tower of $n - 1$ disks on peg C.
2. Move the n-th disk to peg B.

3. Move the upper tower of $n-1$ disks on peg A.
4. Move the n-th disk to peg C.
5. Move the upper tower of $n-1$ disks on peg C.

Let X_n be the number of moves which are necessary to solve the problem: then $X_0 = 0$ and $X_n \leq 3X_{n-1} + 2$ for every $n \geq 0$, and as the nth disk can only be moved when all the others have been moved, the converse inequality must also hold. It is then easy to see, either by induction or by setting $U_n = X_n + 1$, that the unique solution is $X_n = 3^n - 1$.

Exercise 1.3

Show that, in the previous exercise, each legal arrangement of n disks is encountered exactly once.
Solution. There is exactly one legal arrangement per subdivision of the n disks in three (possibly empty) sets. There are $3^n - 1$ moves between displacements, so there are 3^n displacements reached overall. If one of these was touched twice, then it would be possible to reduce the number of moves by performing, the first time we reach said displacement, the chain of steps we would have taken on the second of its occurrences: which contradicts the result we obtained in the previous exercise.

Exercise 1.6

Some of the regions defined by n lines in the plane are infinite, while others are bounded. What is the maximum possible number of bounded regions?

Solution. We know that the n-th line, crossing the previous $n-1$, creates at most n new regions: of those regions, two must be infinite, while the others may be all bounded if no two lines are parallel. Then the maximum number B_n of bounded regions of the planes determined by n satisfies the initial conditions $B_1 = B_2 = 0$ and the recurrence equation $B_n = B_{n-1} + n - 2$. By setting $C_n = B_{n+2}$ we have $C_0 = 0$ and $C_n = C_{n-1} + n$ for every $n \geq 1$: then $C_n = S_n$ for every $n \geq 0$, and $B_n = S_{n-2} = S_n - n - (n-1) = L_n - 2n$.

Exercise 1.9

Consider the following statement:

$$P(n) \ : \ x_1 \cdots x_n \leq \left(\frac{x_1 + \ldots + x_n}{n} \right)^n \quad \forall x_1, \ldots, x_n > 0 \ . \tag{1}$$

This is trivially true for $n = 1$, and is also true for $n = 2$ as $(x_1+x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geq 0$.

1. By setting $x_n = (x_1 + \ldots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ for every $n > 1$.

2. Prove that, for every $n \geq 1$, $P(n)$ and $P(2)$ together imply $P(2n)$.

3. Explain why points 1 and 2 together imply that $P(n)$ is true for every $n \geq 1$.

Solution. Observe that (1) expresses the following, well known fact: the geometric mean of a finite sequence of positive numbers never exceeds their arithmetic mean.

Point 1. Suppose $P(n)$ is true. Then it remains true with the special choice of x_n:

$$\begin{aligned}
& x_1 \cdots x_{n-1} \cdot \frac{x_1 + \ldots + x_{n-1}}{n-1} \\
& \leq \left(\frac{x_1 + \ldots + x_{n-1} + \frac{x_1 + \ldots + x_{n-1}}{n-1}}{n} \right)^n \\
& = \left(\frac{\frac{(n-1)(x_1 + \ldots + x_{n-1}) + (x_1 + \ldots + x_{n-1})}{n-1}}{n} \right)^n \\
& = \left(\frac{x_1 + \ldots + x_{n-1}}{n-1} \right)^n \\
& = \left(\frac{x_1 + \ldots + x_{n-1}}{n-1} \right)^{n-1} \cdot \frac{x_1 + \ldots + x_{n-1}}{n-1} \ .
\end{aligned}$$

As $x_1, \ldots, x_{n-1}$ are arbitrary and $(x_1 + \ldots + x_{n-1})/(n-1) > 0$, $P(n-1)$ is true.

Point 2. Suppose $P(n)$ and $P(2)$ are both true. Then:

$$
\begin{aligned}
& x_1 \cdots x_n \cdot x_{n+1} \cdots x_{2n} \\
& \leq \left(\frac{x_1 + \ldots + x_n}{n} \right)^n \cdot \left(\frac{x_{n+1} + \ldots + x_{2n}}{n} \right)^n \\
& = \left(\left(\frac{x_1 + \ldots + x_n}{n} \right) \cdot \left(\frac{x_{n+1} + \ldots + x_{2n}}{n} \right) \right)^n \\
& \leq \left(\left(\frac{\frac{x_1 + \ldots + x_n}{n} + \frac{x_{n+1} + \ldots + x_{2n}}{n}}{2} \right)^2 \right)^n \\
& = \left(\frac{x_1 + \ldots + x_n + x_{n+1} + \ldots + x_{2n}}{2n} \right)^{2n} \quad .
\end{aligned}
$$

As $x_1, \ldots, x_{2n}$ are arbitrary, $P(2n)$ is true.

Point 3. For every positive integer n there exists an integer $k \geq 0$ and positive integers $m_0 = 2, m_1, \ldots, m_k = n$ such that, for every $i < k$, either $m_{i+1} = 2m_i$ or $m_{i+1} = m_i - 1$. (For instance, set $m_{i+1} = 2m_i$ until $m_i \geq n$, then $m_{i+1} = m_i - 1$ until $m_i = n$.) By points 1 and 2, $P(m_0)$ is true, and $P(m_0)$ and $P(m_i)$ together imply $P(m_{i+1})$ for every $i < k$: therefore, $P(m_k)$ is true. As $m_k = n$ is arbitrary, the thesis follows.