Concrete Mathematics
Exercises from 30 September 2016

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Exercise 1.7

Let $H(n) = J(n+1) - J(n)$. Equation (1.8) tells us that $H(2n) = 2$, and $H(2n+1) = J(2n+2) - J(2n+1) = (2J(n+1)-1) - (2J(n)+1) = 2H(n) - 2$ for every $n \geq 1$. Therefore it seems possible to prove that $H(n) = 2$ for all n, by induction on n. What's wrong here?
Solution. To correctly prove by induction that $H(n) = 2$ for *every* $n \geq 1$, we need to check the induction base for $n = 1$. However, $H(1) = J(2) - J(1) = 1 - 1 = 0$

Exercise 1.8

Solve the recurrence:

$$Q_0 = \alpha \ ; \ \ Q_1 = \beta;$$
$$Q_n = (1 + Q_{n-1})/Q_{n-2} \ , \ \ \text{for } n > 1 \ .$$

Assume that $Q_n \neq 0$ for all $n \geq 0$.
Hint: $Q_4 = (1+\alpha)/\beta$.
Solution. Let us just start computing. We get $Q_2 = (1+\beta)/\alpha$ and $Q_3 =$

$(1+((1+\beta)/\alpha))/\beta = (1+\alpha+\beta)/\alpha\beta$. Then,

$$\begin{aligned} Q_4 &= \frac{1+\frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\ &= \frac{\frac{\alpha\beta+1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\ &= \frac{\frac{(1+\alpha)(1+\beta)}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\ &= \frac{1+\alpha}{\beta}, \end{aligned}$$

and

$$\begin{aligned} Q_5 &= \frac{1+\frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} \\ &= \frac{\frac{\beta+1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} \\ &= = \alpha\ . \end{aligned}$$

Thus, $Q_6 = (1+\alpha)/((1+\alpha)/\beta) = \beta$, and the sequence is periodic.

Important note: The exercise asks us to solve a *second order* recurrence with *two* initial conditions, corresponding to two consecutive indices. To be sure that the solution is a periodic sequence, we must then make sure that *two consecutive values* are repeated.

A note on the repertoire method

Suppose that we have a recursion scheme of the form:

$$\begin{aligned} g(0) &= \alpha\ , \\ g(n+1) &= \Phi(g(n)) + \Psi(n;\beta,\gamma,\ldots) \quad \text{for } \ n \geq 0\ . \end{aligned} \tag{1}$$

Suppose now that:

1. Φ is linear *in* g, *i.e.*, if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
   No hypotheses are made on the dependence of g on n.

2. Ψ is a linear function of the $m-1$ parameters $\beta, \gamma, \ldots$
   No hypotheses are made on the dependence of Ψ on n.

Then the whole system (1) is linear in the parameters $\alpha, \beta, \gamma, \ldots$, *i.e.*, if $g_i(n)$ is the solution corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$, then $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ is the solution corresponding to $\alpha = \lambda_1 \alpha_1 + \lambda_2 \alpha_2, \beta = \lambda_1 \beta_1 + \lambda_2 \beta_2, \gamma = \lambda_1 \gamma_1 + \lambda_2 \gamma_2, \ldots$

We can then look for a general solution of the form

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \ldots \tag{2}$$

i.e., think of $g(n)$ as a linear combination of m functions $A(n), B(n), C(n), \ldots$ according to the coefficients $\alpha, \beta, \gamma, \ldots$

To find these functions, we can reason as follows. Suppose we have a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfying the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$

2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then the functions $A(n), B(n), C(n), \ldots$ are uniquely determined. The reason is that, for every fixed n,

$$\begin{array}{lllll} \alpha_1 A(n) & +\beta_1 B(n) & +\gamma_1 C(n) & +\ldots & = & g_1(n) \\ \vdots & & & & = & \vdots \\ \alpha_m A(n) & +\beta_m B(n) & +\gamma_m C(n) & +\ldots & = & g_m(n) \end{array}$$

is a system of m linear equations in the m unknowns $A(n), B(n), C(n), \ldots$ whose coefficients matrix is invertible.

This general idea can be applied to several different cases. For instance, if the recurrence is second-order:

$$\begin{array}{rcl} g(0) & = & \alpha_0 \, , \\ g(1) & = & \alpha_1 \, , \\ g(n+1) & = & \Phi_0(g(n)) + \Phi_1(g(n-1)) + \Psi(n; \beta, \gamma, \ldots) \quad \text{for } n \geq 1 \, , \end{array} \tag{3}$$

then we will require that Φ_0 and Φ_1 are linear in g, and that Ψ is a linear function of the $m-2$ parameters $\beta, \gamma, \ldots$.

The same can be said of systems of the form:

$$\begin{aligned} g(1) &= \alpha \,, \\ g(kn+j) &= \Phi(g(n)) + \Psi(n; \beta_j, \gamma_j, \dots) \quad \text{for } n \geq 1 \,, \;\; 0 \leq j < k \,. \end{aligned} \tag{4}$$

The previous argument is easily adapted to the new case: this time, the number of tuple-function pairs to determine will be $1 + k \cdot (m-1)$.

For instance, in the Josephus problem we have $k = 2$, $\alpha = 1$, $\Phi(g) = 2g$, $\Psi(n; \beta) = \beta$, $m = 2$, $\beta_0 = -1$, $\beta_1 =$: and we need $3 = 1 + 2 \cdot (2-1)$ tuple-function pairs.

Repertoire method: Exercise 1

Use the repertoire method to solve the following general recurrence:

$$\begin{aligned} g(0) &= \alpha \,, \\ g(n+1) &= 2g(n) + \beta n + \gamma \quad \text{for } n \geq 0 \,. \end{aligned} \tag{5}$$

Solution. The recurrence (5) has the form (1) with $\Phi(g) = 2g$ and $\Psi(n; \beta, \gamma) = \beta n + \gamma$, which are linear in g and in β and γ, respectively: therefore we can apply the repertoire method. The special case $g(n) = 1$ for every $n \geq 0$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, -1)$: thus,

$$A(n) - C(n) = 1 \,.$$

The special case $g(n) = n$ for every $n \geq 0$ corresponds to $(\alpha, \beta, \gamma) = (0, -1, 1)$: thus,

$$-B(n) + C(n) = n \,.$$

The special case $g(n) = 2^n$ for every $n \geq 0$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 0)$: thus,

$$A(n) = 2^n \;\; \text{and consequently} \,, \;\; C(n) = 2^n - 1 \;\; \text{and} \;\; B(n) = 2^n - 1 - n \,.$$

The general solution of (5) is

$$\begin{aligned} g(n) &= \alpha \cdot 2^n + \beta \cdot (2^n - 1 - n) + \gamma \cdot (2^n - 1) \\ &= (\alpha + \beta + \gamma) \cdot 2^n - \beta n - (\beta + \gamma) \,. \end{aligned}$$

Repertoire method: Exercise 2

What if the recurrence (5) had been

$$\begin{aligned} g(0) &= \alpha \, , \\ g(n+1) &= \delta g(n) + \beta n + \gamma \quad \text{for } n \geq 0 \, . \end{aligned} \tag{6}$$

instead?

Solution. The recurrence (6), considered as a family of recurrence equations parameterized by $(\alpha, \beta, \gamma, \delta)$, does *not* have the form (1)! Here, the function Φ depends on both the function g and the parameter δ: because of this, in general $g_1(n) + g_2(n)$ is not the solution for $(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2)$, because $\delta_1 g_1(n) + \delta_2 g_2(n)$ is not, in general, equal to $(\delta_1 + \delta_2)(g_1(n) + g_2(n))$. We cannot therefore use the repertoire method to express $g(n)$ as $g(n) = \alpha \cdot A(n) + \beta \cdot B(n) + \gamma \cdot C(n) + \delta \cdot D(n)$.

However, for every *fixed* δ, (6) *does* have the form (1) with $\Phi(g) = \delta g$ and $\Psi(n; \beta, \gamma) = \beta n + \gamma$: thus, for every fixed δ, we can use the repertoire method to find three functions $A_\delta(n), B_\delta(n), C_\delta(n)$ such that

$$g_\delta(n) = \alpha \cdot A_\delta(n) + \beta \cdot B_\delta(n) + \gamma \cdot C_\delta(n)$$

for every $n \geq 0$. By reasoning as before, the choice $g_\delta(n) = 1$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 1 - \delta)$, thus

$$A_\delta(n) + (1 - \delta) C_\delta(n) = 1 \; : \tag{7}$$

the factor $1 - \delta$ in front of $C_\delta(n)$ rings a bell, and suggests we might have to be careful about the cases $\delta = 1$ and $\delta \neq 1$. Choosing $g_\delta(n) = n$ corresponds to $(\alpha, \beta, \gamma) = (0, 1 - \delta, 1)$, thus

$$(1 - \delta) B_\delta(n) + C_\delta(n) = n \, . \tag{8}$$

We are left with one triple of values to choose. As we had put $g(n) = 2^n$ when $\delta = 2$, we are tempted to just put $g(n) = \delta^n$: but if $\delta = 1$ this would be the same as $g(n) = 1$, which we have already considered. We will then deal separately with the cases $\delta = 1$ and $\delta \neq 1$.

Let us start with the latter. For $\delta \neq 1$ the choice $g_\delta(n) = \delta^n$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 0)$, thus

$$A_\delta(n) = \delta^n \; : \tag{9}$$

by combining this with (7) and (8) we find

$$C_\delta(n) = \frac{1 - A_\delta(n)}{1-\delta} = \frac{1-\delta^n}{1-\delta} = 1 + \delta + \ldots + \delta^{n-1}$$

and

$$B(n) = \frac{n - C_\delta(n)}{1-\delta} = \frac{n - 1 - \delta - \ldots - \delta^{n-1}}{1-\delta} \ .$$

Let us now consider the case $\delta = 1$. Then (7) becomes $A_1(n) = 1$ and (8) becomes $C_1(n) = n$: for the last case, we set $g_1(n) = n^2$, which corresponds to $(\alpha, \beta, \gamma) = (0, 2, 1)$, and find

$$2B_1(n) + C_1(n) = n^2 \ , \tag{10}$$

which yields $B_1(n) = (n^2 - n)/2$.