ITT9131 Concrete Mathematics
Exercises from 29 November

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Exercise 6.4

Express $1 + 1/3 + \ldots + 1/(2n+1)$ in terms of harmonic numbers.
Solution. If the summands $1/2, 1/4, \ldots, 1/2n$ were present, that would be H_{2n+1}. But they are not there, thus their amount—which is $H_n/2$—has been subtracted. In conclusion, $1 + 1/3 + \ldots + 1/(2n+1) = H_{2n+1} - \frac{1}{2}H_n$.

Exercise 6.26

Use summation by parts to evaluate $S_n = \sum_{k=1}^n H_k/k$. *Hint:* Consider also the related sum $\sum_{k=1}^n H_{k-1}/k$.
Solution. As $H_k = H_{k-1} + 1/k$ and $H_0 = 0$, we have $S_n = \sum_{k=1}^n H_{k-1}/k + \sum_{k=1}^n 1/k^2 = T_n + H_n^{(2)}$. We can thus compute S_n by computing T_n.

To use summation by parts, we must write $T_n = \sum_1^{n+1} u(x)\Delta v(x)\delta x$, where u and v are suitable functions. But if we just put $u(k) = H_{k-1}$ and $\Delta v(k) = 1/k$, then $\Delta u(k) = 1/k$ as well, and we can set $v(k) = H_{k-1}$ too, so that $Ev(k) = H_k$ and we may hope to recover S_n on the right-hand side with its sign changed! And indeed,

$$\begin{aligned}
\sum_1^{n+1} u(x)\Delta v(x)\delta x &= u(x)v(x)\big|_{x=1}^{x=n+1} - \sum_1^{n+1} Ev(x)\Delta u(x)\delta x \\
&= (H_{x-1})^2\big|_{x=1}^{x=n+1} - \sum_{k=1}^{n} H_k/k \\
&= H_n^2 - S_n \ :
\end{aligned}$$

then $S_n = T_n + H_n^{(2)} = H_n^2 - S_n + H_n^{(2)}$, which yields $S_n = (H_n^2 + H_n^{(2)})/2$.

Exercise 6.27

Prove the gcd law (6.111) for the Fibonacci numbers.
Solution. We are required to prove that, for positive m and n,

$$\gcd(f_m, f_n) = f_{\gcd(m,n)} \,. \tag{1}$$

We first prove (1) for $m = n+1$. As $f_{n+1} = f_n + f_{n-1}$, a common divisor d of f_{n+1} and f_n should also divide f_{n-1}: then it would also divide $f_{n-2} = f_n - f_{n-1}$, and f_{n-3} as well, and so on up to $f_1 = 1$. Thus, $\gcd(f_{n+1}, f_n) = 1 = f_1 = f_{\gcd(n+1,n)}$.

To prove the general case, we have two different approaches, both exploiting the case $m = n+1$ and the generalized Cassini's identity. We will explore them both, as they both have merits. To fix ideas, we set $m > n \geq 1$.

1. We prove an additional lemma: for every $n, k \in \mathbb{Z}$, f_{kn} is a multiple of f_n. As f_m and f_{-m} differ at most for their sign, it will be sufficient to prove the thesis when k and n are positive integers: we proceed by induction on k, for fixed n. The thesis is surely true for $k = 1$; if it is true for k, then by the generalized Cassini's identity,

$$f_{(k+1)n} = f_{kn+n} = f_n f_{kn+1} + f_{n-1} f_{kn}$$

is a multiple of f_n by inductive hypothesis. As a corollary, for $d = \gcd(m, n)$ we have that both f_m and f_n are multiples of f_d. To complete our proof, we must show that, if $q|f_m$ and $q|f_n$, then $q|f_d$ too: but if we write $d = sm + tn$ as a linear combination with integer coefficients, then

$$f_d = f_{sm+tn} = f_{sm} f_{tn+1} + f_{sm-1} f_{tn}$$

is also divisible by q because of the lemma.

2. By the generalized Cassini identity with $m - n$ in the role of n and n in the role of k,

$$\begin{aligned} \gcd(f_m, f_n) &= \gcd\left(f_n f_{m-n+1} + f_{n-1} f_{m-n}, f_n\right) \\ &= \gcd\left(f_{n-1} f_{m-n}, f_n\right) \\ &= \gcd(f_{m-n}, f_n) \,, \end{aligned}$$

because $\gcd(f_{n-1}, f_n) = 1$ as we had proved before. But we can continue subtracting n until the remainder becomes smaller than n: this happens after $\lfloor m/n \rfloor$ iterations, yielding

$$\gcd(f_m, f_n) = \gcd(f_n, f_{m-\lfloor m/n \rfloor n}) = \gcd(f_n, f_{m \bmod n}) .$$

But the equality above means precisely that we can run the Euclidean algorithm on the indices of the Fibonacci numbers, instead of the Fibonacci numbers themselves! The thesis clearly follows.

Exercise 6.28

For n integer, define the nth *Lucas number* as $L_n = f_{n+1} + f_{n-1}$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	...

1. Use the repertoire method to find the general solution to the recurrence:

$$\begin{aligned} Q_0 &= \alpha \\ Q_1 &= \beta \\ Q_2 &= Q_{n-1} + Q_{n-2} \ \ , \ n > 1 \end{aligned}$$

2. Find a closed form for L_n in terms of ϕ and $\hat{\phi}$.

Solution. **Point 1.** The Fibonacci numbers satisfy the recurrence with $\alpha = 0$, $\beta = 1$. The Lucas numbers also satisfy the recurrence:

$$L_{n-1} + L_{n-2} = f_n + f_{n-2} + f_{n-1} + f_{n-3} = f_{n+1} + f_{n-1} = L_n$$

We thus only need to reconstruct the initial condition through the system

$$\begin{aligned} x f_0 + y L_0 &= \alpha \\ x f_1 + y L_1 &= \beta \end{aligned}$$

that is,

$$\begin{aligned} 2y &= \alpha \\ x + y &= \beta \end{aligned}$$

which yields $y = \alpha/2$, $x = \beta - \alpha/2$. Therefore,

$$Q_n = xf_n + yL_n = \frac{\alpha}{2}(L_n - f_n) + \beta f_n$$

Point 2. We know that $f_n = (\phi^n - \hat{\phi}^n)/\sqrt{5}$. Then

$$\begin{aligned} L_n &= \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} \\ &= \frac{\phi^{n-1}(\phi^2 + 1) - \hat{\phi}^{n-1}(\hat{\phi}^2 + 1)}{\sqrt{5}} \end{aligned}$$

But

$$\phi^2 + 1 = \frac{6 + 2\sqrt{5}}{4} + 1 = \frac{5 + \sqrt{5}}{2} = \phi\sqrt{5}$$

and similarly,

$$\hat{\phi}^2 + 1 = \frac{6 - 2\sqrt{5}}{4} + 1 = \frac{5 - \sqrt{5}}{2} = -\hat{\phi}\sqrt{5}$$

Consequently,

$$L_n = \frac{\phi^{n-1}(\phi\sqrt{5}) - \hat{\phi}^{n-1}(-\hat{\phi}\sqrt{5})}{\sqrt{5}} = \phi^n + \hat{\phi}^n\,.$$