Concrete Mathematics
Exercises from Chapter 7

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Warmups

Exercise 7.7

Solve the recurrence:

$$\begin{aligned} g_0 &= 1 \\ g_n &= g_{n-1} + 2g_{n-2} + \ldots + ng_0 \end{aligned}$$

Solution. Let $G(z)$ be the generating function of the sequence $\langle g_0, g_1, g_2, \ldots \rangle$. The recurrence above tells us that $G(z)$ is the convolution of itself with the generating function of the sequence $\langle 0, 1, 2, \ldots \rangle$, which is $z/(1-z)^2$: except for the first term, which is 1 instead of $0 = 0 \cdot g_0$. Hence,

$$G(z) = 1 + \frac{zG(z)}{(1-z)^2}$$

which rewrites as

$$(1-z)^2 G(z) = (1-z)^2 + zG(z)$$

which yields

$$(1 - 3z + z^2)G(z) = (1-z)^2$$

that is,

$$G(z) = \frac{1 - 2z + z^2}{1 - 3z + z^2} = 1 + \frac{z}{1 - 3z + z^2}$$

The first summand on the right-hand side is clearly the generating function of the sequence $a_n = [n = 0]$; the second one, is the generating function of $b_n = f_{2n}$, where f_n is the nth Fibonacci number. Therefore, $g_n = f_{2n} + [n = 0]$.

An exercise by Albert R. Meyer and Ronitt Rubinfeld

Let a_n be the number of string on a ternary alphabet that contain a double character, *i.e.*, a sequence xx with x a letter.

1. Find a recurrence for a_n.

2. Let $G(z)$ be the generating function of the sequence $\langle a_0, a_1, a_2, \ldots \rangle$. Prove that
$$G(z) = \frac{-z}{1-2z} + \frac{z}{(1-2z)(1-3z)} \tag{1}$$

3. Find r and s such that
$$\frac{1}{(1-2z)(1-3z)} = \frac{r}{1-2z} + \frac{s}{1-3z} \tag{2}$$

4. Find a closed form for a_n.

Solution.

Point 1. Call "good" a sequence with a double letter, "bad" a sequence without. Then a_n is the number of good sequences of n letters.

There are no good sequences of length 0 or 1, so $a_0 = a_1 = 0$. For $n \geq 2$, a good sequence of length n can be obtained from a sequence of length $n-1$ in two ways: Either we add any letter at the end of a good sequence, or we duplicate the last letter of a bad sequence. No string allows applying both methods, so the number of good strings of length n is three times the number of good strings of length $n-1$, plus the number of bad strings of length $n-1$. Thus,
$$a_n = 3a_{n-1} + (3^{n-1} - a_{n-1}) = 2a_{n-1} + 3^{n-1} \tag{3}$$
for every $n \geq 2$.

Point 2. We want to rewrite (3) in terms of generating functions. Since the recurrence only holds for $n \geq 2$, we must consider formal power series whose constant and linear term are zero. Recall that $1/(1-\alpha z)$ is the generating function of the sequence of the powers of α.

Let $G(z)$ be the generating function of $\langle a_0, a_1, a_2, a_3, \ldots \rangle$. Then $zG(z)$ is the generating function of $\langle 0, a_0, a_1, a_2, \ldots \rangle$, while that of $\langle 0, 1, 3, 9, 27, \ldots \rangle$ is $z/(1-3z)$. Therefore, in terms of generating functions, (3) is rewritten as
$$G(z) - a_0 - a_1 z = 2z(G(z) - a_0) + z\left(\frac{1}{1-3z} - 1\right) :$$

which, since $a_0 = a_1 = 0$, gives

$$(1-2z)G(z) = \frac{z}{1-3z} - z$$

which yields (1).

Point 3. Equation (2) is satisfied if and only if $r\cdot(1-3z)+s\cdot(1-2z) = 1$ whatever z is. For $z = 1/2$ we find $r \cdot (1-3/2) = 1$, so $r = -2$. For $z = 1/3$ we find $s \cdot (1-2/3) = 1$, so $s = 3$. In conclusion,

$$\frac{1}{(1-2z)(1-3z)} = \frac{3}{1-3z} - \frac{2}{1-2z}\,.$$

Point 4. We can rewrite (1) as follows:

$$\begin{aligned} G(z) &= \frac{-z(1-3z)+z}{(1-2z)(1-3z)} \\ &= \frac{-z+3z^2+z}{(1-2z)(1-3z)} \\ &= 3z^2 \cdot \left(\frac{3}{1-3z} - \frac{2}{1-2z}\right) \end{aligned}$$

For $n \geq 2$ it must then be:

$$\begin{aligned} a_n &= [z^n]G(z) \\ &= 3 \cdot \left([z^{n-2}]\left(\frac{3}{1-3z}\right) - [z^{n-2}]\left(\frac{2}{1-2z}\right)\right) \\ &= 9 \cdot 3^{n-2} - 6 \cdot 2^{n-2} \\ &= 3 \cdot (3^{n-1} - 2^{n-1})\,. \end{aligned}$$

Exercise 7.11

Let $a_n = b_n = c_n = 0$ for $n < 0$, and

$$A(z) = \sum_n a_n z^n \;\; ; \;\; B(z) = \sum_n b_n z^n \;\; ; \;\; C(z) = \sum_n c_n z^n$$

1. Express C in terms of A and B when $c_n = \sum_{j+2k \leq n} a_j b_k$.

2. Express A in terms of B when $nb_n = \sum_{k=0}^{n} 2^k a_k/(n-k)!$

3. Express A in terms of B when $a_n = \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k}$. Then, construct $\{f_n(r)\}_{n\geq 0}$ such that $b_n = \sum_{k=0}^{n} f_k(r) a_{n-k}$.

Solution.
Point 1. We know that, if $a_n = [z^n]G(z)$, then $\sum_{k\leq n} a_k = [z^n]\frac{G(z)}{1-z}$. Then we can solve point 1 as soon as we find $G(z)$ such that $[z^n]G(z) = \sum_{j+2k=n} a_j b_k$. But the latter is the coefficient of index n of the convolution of A with a power series whose odd-indexed coefficients are 0, and whose coefficient of index $2k$ is b_k: such function is precisely $B(z^2)$. Therefore,

$$C(z) = \frac{A(z)B(z^2)}{1-z}$$

We will leave the next points for the next lecture.