ITT9131 Concrete Mathematics
Solutions to final exam of 3 January 2017

Revision: 3 January 2017

Exercise 1

(12 points) Solve the recurrence:

$$\begin{aligned} g_0 &= 0\,; & g_1 &= 2\,; \\ g_n &= \frac{5}{2}\,g_{n-1} - g_{n-2} & \forall n &\geq 2\,. \end{aligned} \tag{1}$$

Solution:

The recurrence (1) is easily solved with generating functions via the Rational Expansion Theorem. Let us follow the method step by step:

1. We rewrite (1) so that it holds for every $n \in \mathbb{Z}$, with the convention that $g_n = 0$ if $n < 0$. We have to check the cases $n = 0$ and $n = 1$:

   - For $n = 0$ we have $g_0 = 0$ and $\frac{5}{2}g_{-1} - g_{-2} = 0$. Thus, we need no correction summand.
   - For $n = 1$ we have $g_1 = 2$ but $\frac{5}{2}g_0 - g_{-1} = 0$. Thus, we need to add a correction summand 2.

   The equation (1) rewritten for arbitrary $n \in \mathbb{Z}$ becomes:

$$g_n = \frac{5}{2}g_{n-1} - g_{n-2} + 2\,[n = 1] \quad \forall n \in \mathbb{Z}\,. \tag{2}$$

2. Let $G(z) = \sum_n g_n z^n$ be the generating function of the sequence $\langle g_n \rangle$. By multiplying (2) by z^n for every $n \in \mathbb{Z}$ and summing over n we obtain:

$$\begin{aligned} G(z) &= \sum_n g_n z^n \\ &= \frac{5}{2} \sum_n g_{n-1} z_n - \sum_n g_{n-2} z^n + 2 \sum_n [n = 1]\, z^n \\ &= \frac{5}{2} \sum_n g_n z^{n+1} - \sum_n g_n z^{n+2} + 2z \\ &= \frac{5}{2}\, z\, G(z) - z^2\, G(z) + 2z\,. \end{aligned}$$

3. By solving the above with respect to $G(z)$ we get

$$G(z) \cdot \left(1 - \frac{5}{2} z + z^2\right) = 2z\,,$$

which yields

$$G(z) = \frac{2z}{1 - \frac{5}{2} z + z^2}\,. \tag{3}$$

4. Equation (3) has the form $G(z) = P(z)/Q(z)$ where $P(z) = 2z$ and $Q(z) = 1 - \frac{5}{2} z + z^2 = (1 - 2z)(1 - z/2)$. We can then apply the Rational Expansion Theorem with $\rho_1 = 2$, $\rho_2 = 1/2$, and $d_1 = d_2 = 1$. As $Q'(z) = 2z - \frac{5}{2}$, we find

$$a_1 = \frac{(-2) \cdot (2 \cdot 1/2)}{2 \cdot 1/2 - 5/2} = \frac{-2}{-3/2} = \frac{4}{3}$$

and

$$a_2 = \frac{(-1/2) \cdot (2 \cdot 2)}{2 \cdot 2 - 5/2} = \frac{-2}{3/2} = -\frac{4}{3}\,.$$

We can thus conclude:

$$g_n = \frac{4}{3} \left(2^n - \frac{1}{2^n}\right)\,.$$

Exercise 2

(10 points) For $n \in \mathbb{N}$ and $r, s \in \mathbb{R}$ compute

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{r+k}{k} \binom{s}{n-k} .$$

Solution: The sequence $\langle S_n \rangle$ is the convolution of the sequences $\left\langle (-1)^n \binom{r+n}{n} \right\rangle$ and $\left\langle \binom{s}{n} \right\rangle$. We know that $\sum_{n \geq 0} \binom{r+n}{n} z^n = 1/(1-z)^{r+1}$ and $\sum_{n \geq 0} \binom{s}{n} = (1+z)^s$: by replacing z with $-z$ in the first power series, we obtain

$$\sum_{n \geq 0} (-1)^n \binom{r+n}{n} z^n = \frac{1}{(1+z)^{r+1}} .$$

(Alternatively, since $c^{\underline{n}} = (-1)^n (-c)^{\overline{n}} = (-1)^n (n-1-c)^{\underline{n}}$ for every $c \in \mathbb{R}$, by putting $c = r+n$ we obtain $(-1)^n \binom{r+n}{n} = \binom{-r-1}{n}$, which yields the same generating function.) Then the generating function of $\langle S_n \rangle$ is

$$S(z) = \frac{(1+z)^s}{(1+z)^{r+1}} = (1+z)^{s-r-1} \ :$$

it follows immediately that

$$\sum_{k=0}^{n} (-1)^k \binom{r+k}{k} \binom{s}{n-k} = \binom{s-r-1}{n} .$$

Exercise 3

(8 points) Determine for which integer values of n the number $n^{13} - 2n^7 + n$ is divisible by 98.
Solution: As $98 = 2 \cdot 7^2$ as a product of powers of primes, we must show that $n^{13} - 2n^7 + n$ is divisible by both 2 and 49. One part is easy: the second summand is even, and the other two are either both even or both odd, so the sum is even. For the other part, we factor the polynomial and obtain:

$$n^{13} - 2n^7 + n = n \cdot (n^{12} - 2n^6 + 1) = n \cdot (n^6 - 1)^2 .$$

If n is not a multiple of 7, then $n^6 - 1$ is by Fermat's little theorem, and as there are two such factors, $n^{13} - 2n^7 + n$ is indeed divisible by 49. If n is a multiple of 7, however, then $n^6 - 1$ is not, and since we only have one factor n, it must be n that is divisible by 49.

In conclusion, $n^{13} - 2n^7 + n$ is divisible by 98 if and only if n is either divisible by 49, or not divisible by 7.

Questions

1. How many regions can be obtained, at most, by splitting the plane with 12 straight lines?
   $\frac{12 \cdot 13}{2} + 1 = 79$, *according to the formula from the first chapter.*

2. How do we compute a sum S_n by the perturbation method?
   If $S_n = \sum_{k=0}^{n} a_k$, *we write* S_{n+1} *on the one hand as* $S_n + a_{n+1}$, *and on the other hand as* $a_0 + \sum_{k=1}^{n+1} a_k$; *we rewrite the new sum as a function of* S_n; *and we solve with respect to* S_n.

3. Write a function $u(x)$ such that $\Delta u(x) = 3^x$.
   $u(x) = \frac{1}{2} \cdot 3^x$ *is one such function.*

4. Write the definition of the sum of a sequence of nonnegative numbers.
   For "sequence", we mean "infinite sequence". If $a_k \geq 0$ *for every* $k \geq 0$, *then* $\sum_{k \geq 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k$ *and* $\sum_{k \geq 0} a_k = \sup_{K \subseteq \mathbb{N} \text{ finite}} \sum_{k \in K} a_k$ *are equivalent definitions: any one of the two is considered a valid answer.*

5. State the generalized pigeonhole principle.
   Let m *and* n *be positive integers. If* n *items are to be put in* m *boxes, then at least one of the boxes will contain no less than* $\lceil n/m \rceil$ *objects, and at least one of the boxes will contain no more than* $\lfloor n/m \rfloor$ *objects.*

6. State the fundamental theorem of arithmetics.
   Every integer greater than 1 can be written in a unique way as a product of powers of prime numbers.

7. What is a Carmichael number?
   A Carmichael number is a composite positive integer c *such that* $a^{c-1} \equiv 1 \pmod{c}$ *for every integer* $1 < a < c$ *such that* $\gcd(a, c) = 1$.

8. What is Euler's totient function?
It is the function ϕ, defined on positive integers, such that $\phi(m)$ is the number of those positive integers $1 \leq x \leq m$ such that $\gcd(x, m) = 1$. The definition with $0 < x < m$ is also valid, because $\gcd(m, m) = \gcd(0, m) = m$.

9. Is $13^{56} - 1$ divisible by 29?
Yes, because $13^{56} - 1 = (13^{28} - 1) \cdot (13^{28} + 1)$, and as 13 and 29 are both prime, $13^{28} - 1$ is divisible by 29 by Fermat's little theorem.

10. State the hexagon property.
For every $n \geq 2$ and $0 < k < n$,

$$\binom{n-1}{k-1} \cdot \binom{n}{k+1} \cdot \binom{n+1}{k} = \binom{n-1}{k} \cdot \binom{n}{k-1} \cdot \binom{n+1}{k+1}.$$

The following, more informal answer is also valid: Given a nontrivial binomial coefficient $\binom{n}{k}$ with n and k positive, and considering the hexagon around it determined by the six binomial coefficients around it in Pascal's triangle, the product of the first, third, and fifth vertex of the hexagon equals that of the second, fourth, and sixth.

11. How many ways are there of arranging 6 objects into 2 nonempty subsets?
Using the Stirling numbers of the second kind: $\left\{ \begin{matrix} 6 \\ 2 \end{matrix} \right\} = 2^5 - 1 = 31.$

12. Write the recurrence equation for the Stirling numbers of the first kind.

$$\left[\begin{matrix} n+1 \\ k \end{matrix} \right] = (n-1) \left[\begin{matrix} n \\ k \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right].$$

13. Write Cassini's identity.
$f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ *for every $n \in \mathbb{Z}$, where f_n is the nth Fibonacci number.*

14. What is the convergence radius of a power series?
Let $\sum_{n \geq 0} a_n z^n$ be the power series. Any of the following answers is valid:

- *The convergence radius is the value $R \in [0, \infty]$ defined by*
$$\frac{1}{R} = \limsup_{n \geq 0} \sqrt[n]{|a_n|}\,,$$
with the conventions $1/0 = \infty$ and $1/\infty = 0$.
- *The convergence radius is the unique value $R \in [0, \infty]$ such that the power series converges at every z such that $|z| < R$, and does not converge at any z such that $|z| > R$.*

15. Let $0 \leq r < m$ be integers. What is the generating function of the sequence $\langle [n \equiv r \pmod{m}] \rangle$?
$\dfrac{z^r}{1 - z^m}$*: the sequence is the translation by r positions to the right of $\langle [m|n] \rangle$, which we know to have the generating function $\dfrac{1}{1 - z^m}$.*

16. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\langle g_n\,[n\,\text{even}] \rangle$?
$\dfrac{G(z) + G(-z)}{2}$*, according to the formula from Chapter 7.*

17. What is the generating function of the sequence $\left\langle \frac{1}{(n+1)!}\,[n \geq 0] \right\rangle$?
$\dfrac{e^z - 1}{z}$*. When shifting to the left by one position, we must be sure that we still obtain a power series when we divide by z, which is done by putting to zero the constant term.*

18. For $m \geq 0$ write the generating function of the sequence $\left\langle \begin{bmatrix} m \\ n \end{bmatrix} \right\rangle$.
$\sum_{n \geq 0} \begin{bmatrix} m \\ n \end{bmatrix} z^n = z^{\overline{m}}$*, according to the formula from Chapter 6.*

19. How many ways are there to arrange correctly 6 pairs of parentheses?
The number of such ways is the Catalan number of index 6:
$$C_6 = \frac{1}{6+1}\binom{12}{6} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{6!} = 132\,.$$

20. What is the exponential generating function of the sequence $\langle g_n \rangle$?
It is the generating function of the sequence $\langle g_n / n! \rangle$.