ITT9131 – Concrete Mathematics
Solutions to the midterm exam

Silvio Capobianco

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Exercise 1

(10 points) Solve the recurrence:

$$\begin{aligned} T_0 &= 1\,; \\ T_n &= 2T_{n-1} + \left(\frac{3}{2}\right)^n + 2^n H_n \quad \forall n \geq 1\,. \end{aligned} \tag{1}$$

Solution: The system (1) has the form

$$\begin{aligned} a_0 T_0 &= 1\,; \\ a_n T_n &= b_n T_{n-1} + c_n \quad \forall n \geq 1 \end{aligned}$$

with

$$a_n = 1\ ;\ b_n = 2\ ;\ c_n = \left(\frac{3}{2}\right)^n + H_n\,.$$

This suggests using a summation factor:

$$s_0 = 1\ ;\ s_n = \prod_{j=1}^{n} \frac{a_{j-1}}{b_j} = \frac{1}{2^n} \ \ \forall n \geq 1\,.$$

Then, by putting $U_n = s_n a_n T_n = T_n/2^n$ and simplifying, we get

$$\begin{aligned} U_0 &= 1\,; \\ U_n &= U_{n-1} + \left(\frac{3}{4}\right)^n + H_n \quad \forall n \geq 1\ : \end{aligned}$$

which clearly has the solution

$$\begin{aligned}
U_n &= 1 + \sum_{k=1}^{n} \left(\left(\frac{3}{4} \right)^k + H_k \right) \\
&= \sum_{k=0}^{n} \left(\frac{3}{4} \right)^k + \sum_{k=1}^{n} H_k \\
&= \frac{4^{n+1} - 3^{n+1}}{4^n} + (n+1)H_n - n
\end{aligned}$$

In the end, the solution to (1) is:

$$T_n = \frac{4^{n+1} - 3^{n+1}}{2^n} + 2^n \cdot ((n+1)H_n - n) \ .$$

Exercise 2

(8 points) Prove that $n^{21} - n^{19} - n^3 + n$ is divisible by 114 for every integer $n \geq 1$.

Solution: As $114 = 2 \cdot 3 \cdot 19$ as a product of (powers of) primes, we must prove that $n^{21} - n^{19} - n^3 + n$ is divisible by 2, 3, and 19 for every $n \geq 1$. Factoring the polynomial, we get:

$$n^{21} - n^{19} - n^3 + n = n \cdot (n^{20} - n^{18} - n^2 + 1) = n \cdot (n^{18} - 1) \cdot (n^2 - 1) \, .$$

This decomposition tells us that $n^{21} - n^{19} - n^3 + n$ is divisible by $n^{19} - n$, which in turn is divisible by 19 because of Fermat's last theorem. Moreover, as $n^2 - 1 = (n-1)(n+1)$, the number $n^{21} - n^{19} - n^3 + n$ always has the three consecutive factors $n - 1$, n, and $n + 1$: of those, exactly one is a multiple of 3, and at least one is even.

Exercise 3

1. (3 points.) Prove that, for every $n \geq 1$,

$$H_n \leq 1 + \lfloor \lg n \rfloor \ , \tag{2}$$

   where lg is the base-2 logarithm.

2. (9 points.) Use the inequality (2) to evaluate the infinite sum:

$$\sum_{k \geq 1} k^{\underline{-2}} H_k \,. \tag{3}$$

Important: Point 2 can be solved without having solved point 1, as it only asks to use the inequality (2), not to have proven it.

Solution: For $n \geq 1$ let $m = \lfloor \lg n \rfloor$, so that $2^m \leq n \leq 2^{m+1} - 1$. Then

$$\begin{aligned} H_n &\leq \sum_{k=1}^{2^{m+1}-1} \frac{1}{k} \\ &= \sum_{j=0}^{m} \sum_{k=2^j}^{2^{j+1}-1} \frac{1}{k} \\ &\leq \sum_{j=0}^{m} \sum_{k=2^j}^{2^{j+1}-1} \frac{1}{2^j} \\ &= \sum_{j=0}^{m} 1 = m + 1 = 1 + \lfloor \lg n \rfloor \ . \end{aligned}$$

Let now $u(x) = H_x$ and $v(x) = -x^{\underline{-1}} = -1/(x+1)$, so that $\Delta u(x) = \frac{1}{x+1} = x^{\underline{-1}}$ and $\Delta v(x) = x^{\underline{-2}}$. Then for every $n \geq 2$:

$$\begin{aligned} \sum_{1 \leq k < n} k^{\underline{-2}} H_k &= \sum_{1}^{n} u(x)\, \Delta v(x)\, \delta x \\ &= -x^{\underline{-1}} H_x \Big|_1^n - \sum_{1}^{n} E v(x)\, \Delta u(x)\, \delta x \\ &= -\frac{1}{n+1} \cdot H_n + \frac{1}{2} + \sum_{1}^{n} (x+1)^{\underline{-1}} x^{\underline{-1}}\, \delta x \\ &= \frac{1}{2} - \frac{H_n}{n+1} + \sum_{1}^{n} x^{\underline{-2}}\, \delta x \\ &= \frac{1}{2} - \frac{H_n}{n+1} - x^{\underline{-2}} \Big|_1^n \\ &= \frac{1}{2} - \frac{H_n}{n+1} - \frac{1}{n+1} + \frac{1}{2} \\ &= 1 - \frac{H_n + 1}{n+1} \,. \end{aligned}$$

Because of the inequality (2), the second summand vanishes for $n \to \infty$. We can then conclude that:

$$\sum_{k\geq 1} k^{-2}\, H_k = 1\,.$$