ITT9130 – Concrete Mathematics
November 11, 2014: Midterm exam

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Exercise 1

Prove that, for every nonnegative integer n, the quantity

$$n^{16} - n^{14} - n^4 + n^2$$

is divisible by 78.

Solution

By straightforward factorization,

$$\begin{aligned} n^{16} - n^{14} - n^4 + n^2 &= n^{12} \cdot (n^4 - n^2) - (n^4 - n^2) \\ &= (n^{12} - 1) \cdot (n^4 - n^2) \\ &= (n^{12} - 1) \cdot n^2 \cdot (n^2 - 1) \\ &= (n^{13} - n) \cdot ((n-1) \cdot n \cdot (n+1)) \end{aligned}$$

The product $(n-1) \cdot n \cdot (n+1)$ is made of three consecutive factors, of which one is divisible by 3, and at least one is even: the product itself is thus divisible by 6. The product $n^{13} - n$ is divisible by 13 because of Fermat's little theorem. We may thus conclude that, for every $n \geq 0$, $n^{16} - n^{14} - n^4 + n^2$ is divisible by $6 \cdot 13 = 78$.

Exercise 2

Solve the following recurrence:

$$\begin{aligned} T_0 &= 1 \; , \\ T_n &= -nT_{n-1} + 3 \cdot n \cdot n! \quad \text{for} \;\; n > 0 \; . \end{aligned} \tag{1}$$

Hint: consider the general solution to the recurrence:

$$\begin{aligned} U_0 &= \alpha \; , \\ U_n &= U_{n-1} + (-1)^n \cdot (\beta n + \gamma) \quad \text{for} \;\; n > 0 \; . \end{aligned} \tag{2}$$

Solution

If we write $T_n = (-1)^n n! \, U_n$, then (1) becomes

$$\begin{aligned} U_0 &= 1 \; , \\ (-1)^n n! U_n &= -n(-1)^{n-1}(n-1)! U_{n-1} + 3 \cdot n \cdot n! \quad \text{for} \;\; n > 0 \;\; : \end{aligned}$$

we divide the second equation by $(-1)^n n!$ and get

$$\begin{aligned} U_0 &= 1 \; , \\ U_n &= U_{n-1} + (-1)^n \cdot 3n \quad \text{for} \;\; n > 0 \; , \end{aligned}$$

which in turn is a special case of (2) for $\alpha = 1$, $\beta = 3$, $\gamma = 0$. So let's search for a solution of (2) of the form:

$$U_n = \alpha \cdot A(n) + \beta \cdot B(n) + \gamma \cdot C(n) \; .$$

The choice $U_n = 1$ for every $n \geq 0$ corresponds to $\alpha = 1, \beta = \gamma = 0$: thus, $A(n) = 1$. The choice $U_n = (-1)^n$ corresponds to $\alpha = 1, \beta = 0, \gamma = 2$: thus, $A(n) + 2C(n) = (-1)^n$, from which follows

$$C(n) = \frac{((-1)^n - 1)}{2} = -[n \, \text{is} \, \text{odd}] \; .$$

The choice $U_n = (-1)^n \cdot n$ corresponds to $\alpha = 0, \beta = 2, \gamma = -1$: which yields $2B(n) - C(n) = (-1)^n \cdot n$, and consequently,

$$B(n) = \frac{(-1)^n n - [n \, \text{is} \, \text{odd}]}{2} = (-1)^n \cdot \left\lceil \frac{n}{2} \right\rceil \; .$$

Finally, for $\alpha = 1, \beta = 3, \gamma = 0$ we have $U_n = 1 + 3 \cdot (-1)^n \cdot \lceil n/2 \rceil$: thus,

$$T_n = (-1)^n \cdot n! \cdot \left(1 + 3 \cdot (-1)^n \cdot \left\lceil \frac{n}{2} \right\rceil \right) = \left(3 \left\lceil \frac{n}{2} \right\rceil + (-1)^n \right) \cdot n! \; .$$

Exercise 3

For $n \geq 0$, evaluate

$$S_n = \sum_{0 \leq k < n} k(k-1)H_k \,.$$

Solution

The generic summand can be rewritten as $k^{\underline{2}}H_k$: this suggests using summation by parts with $u(x) = H_x$ and $\Delta v(x) = x^{\underline{2}}$ Indeed, with this choice we get $\Delta u(x) = 1/(x+1)$ and $v(x) = x^{\underline{3}}/3$, so that

$$Ev(x)\Delta u(x) = \frac{(x+1)^{\underline{3}}}{3} \cdot \frac{1}{x+1} = \frac{x^{\underline{2}}}{3} \,.$$

Let us proceed:

$$\begin{aligned}
\sum_{0 \leq k < n} k(k-1)H_k &= \sum_0^n x^{\underline{2}} H_x \, \delta(x) \\
&= \left[\frac{x^{\underline{3}}}{3} H_x \right]_0^n - \frac{1}{3} \sum_0^n x^{\underline{2}} \, \delta(x) \\
&= \frac{n^{\underline{3}}}{3} H_n - 0 - \frac{1}{3} \left[\frac{x^{\underline{3}}}{3} \right]_0^n \\
&= \frac{n^{\underline{3}}}{3} H_n - \frac{1}{3} \left(\frac{n^{\underline{3}}}{3} - 0 \right) \\
&= \frac{n^{\underline{3}}}{3} \left(H_n - \frac{1}{3} \right) \,.
\end{aligned}$$