ITT9132 Concrete Mathematics
Exercises from Week 2

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Exercise 1.2

Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed. Here, A is the start peg, B the stop peg, and C the spool peg.

Solution. For $n = 1$ the shortest sequence is $A \to C$, $C \to B$. For $n = 2$ it is:

1. $A \to C$.
2. $C \to B$.
3. $A \to C$.
4. $B \to C$. Note that the whole tower is on peg C now.
5. $C \to A$.
6. $C \to B$.
7. $A \to C$.
8. $C \to B$.

For the general case, observe that the strategy that solves the problem for n disks works as follows:

1. Move the upper tower of $n - 1$ disks on peg B.

2. Move the n-th disk to peg C.
3. Move the upper tower of $n-1$ disks on peg A.
4. Move the n-th disk to peg B.
5. Move the upper tower of $n-1$ disks on peg B.

Then the number X_n of moves needed by the strategy to solve the problem with n disks satisfies $X_0 = 0$ and $X_n = 3X_{n-1} + 2$ for every $n > 0$. It is easy to see that the only solution is $X_n = 3^n - 1$.

Exercise 1.3

Show that, in the previous exercise, each legal arrangement of n disks is encountered exactly once.

Solution. There is exactly one legal arrangement per subdivision of the n disks in three (possibly empty) sets. There are $3^n - 1$ moves between displacements, so there are 3^n displacements reached overall. If one of these was touched twice, then it would be possible to reduce the number of moves by performing, the first time we reach said displacement, the chain of steps we would have taken on the second of its occurrences: which contradicts the result we obtained in the previous exercise.

Exercise 1.4

Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, according to Lucas's original rules?

Solution. By contradiction, let n be the smallest number of tiles such that there are two configurations A and B of n tiles which are at least 2^n moves apart. Then the largest tile in A and B must be on two different pegs, otherwise A could be turned into B by only moving the $n-1$ smaller tiles, which requires less than 2^{n-1} moves by our hypothesis on n. But then, the problem can be solved by first transforming A into some other configuration C where only the $n-1$ smaller tiles are moved, then moving the larger tile, and finally transforming C into B by only moving the $n-1$ smaller pegs: by our hypothesis on n, this requires at most $(2^{n-1}-1)+1+(2^{n-1}-1) = 2^n - 1$ moves. This is a contradiction.

The technique of minimum counterexample

The proof technique we employed to solve the previous exercise is based on the following, intuitive[1] fact:

Every nonempty set of natural numbers has a minimum.

Suppose that we have a sequence $\{P(n)\}_{n \geq 0}$ of propositions depending on natural numbers, and that we want to prove that they are all true. To do this, we may reason by contradiction and suppose that they are *not* all true: then the set $F = \{n \geq 0 \mid P(n) \,\text{is false}\}$ is nonempty, and has a minimum $\bar{n}$. From here, we derive a contradiction. Some possibilities are:

1. Prove that $P(\bar{n})$ is true.
2. Prove that $P(m)$ is false for some $m < \bar{n}$).

This technique, or variants of it, also works with other kinds of induction such as *structural induction.*

Exercise 1.7

Let $H(n) = J(n+1) - J(n)$. Equation (1.8) tells us that $H(2n) = 2$, and $H(2n+1) = J(2n+2) - J(2n+1) = (2J(n+1)-1) - (2J(n)+1) = 2H(n) - 2$ for every $n \geq 1$. Therefore it seems possible to prove that $H(n) = 2$ for all n, by induction on n. What's wrong here?

Solution. To correctly prove by induction that $H(n) = 2$ for every $n \geq 1$, we need to check the induction base for $n = 1$. However, $H(1) = J(2) - J(1) = 1 - 1 = 0$.

Exercise 1.9

Consider the following statement:

$$P(n) \ : \ x_1 \cdots x_n \leq \left(\frac{x_1 + \ldots + x_n}{n} \right)^n \quad \forall x_1, \ldots, x_n > 0 \, . \tag{1}$$

This is trivially true for $n = 1$, and is also true for $n = 2$ as $(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geq 0$.

[1] Actually, it is equivalent to induction.

1. By setting $x_n = (x_1 + \ldots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ for every $n > 1$.
2. Prove that, for every $n \geq 1$, $P(n)$ and $P(2)$ together imply $P(2n)$.
3. Explain why points 1 and 2 together imply that $P(n)$ is true for every $n \geq 1$.

Solution. Observe that (1) expresses the following, well known fact: the geometric mean of a finite sequence of positive numbers never exceeds their arithmetic mean. The formula is true for $n = 2$, because $\sqrt{x_1 x_2} \leq \frac{x_1 + x_2}{2}$ is equivalent to $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$.

Point 1. Suppose $P(n)$ is true. Then it remains true with the special choice of x_n:

$$\begin{aligned}
x_1 \cdots x_{n-1} \cdot \frac{x_1 + \ldots + x_{n-1}}{n-1} &\leq \left(\frac{x_1 + \ldots + x_{n-1} + \frac{x_1 + \ldots + x_{n-1}}{n-1}}{n} \right)^n \\
&= \left(\frac{\frac{(n-1)(x_1 + \ldots + x_{n-1}) + (x_1 + \ldots + x_{n-1})}{n-1}}{n} \right)^n \\
&= \left(\frac{x_1 + \ldots + x_{n-1}}{n-1} \right)^n \\
&= \left(\frac{x_1 + \ldots + x_{n-1}}{n-1} \right)^{n-1} \cdot \frac{x_1 + \ldots + x_{n-1}}{n-1} .
\end{aligned}$$

As $x_1, \ldots, x_{n-1}$ are arbitrary and $(x_1 + \ldots + x_{n-1})/(n-1) > 0$, $P(n-1)$ is true.

Point 2. Suppose $P(n)$ and $P(2)$ are both true. Then:

$$\begin{aligned}
x_1 \cdots x_n \cdot x_{n+1} \cdots x_{2n} &\leq \left(\frac{x_1 + \ldots + x_n}{n} \right)^n \cdot \left(\frac{x_{n+1} + \ldots + x_{2n}}{n} \right)^n \\
&= \left(\left(\frac{x_1 + \ldots + x_n}{n} \right) \cdot \left(\frac{x_{n+1} + \ldots + x_{2n}}{n} \right) \right)^n \\
&\leq \left(\left(\frac{\frac{x_1 + \ldots + x_n}{n} + \frac{x_{n+1} + \ldots + x_{2n}}{n}}{2} \right)^2 \right)^n \\
&= \left(\frac{x_1 + \ldots + x_n + x_{n+1} + \ldots + x_{2n}}{2n} \right)^{2n} .
\end{aligned}$$

As $x_1, \ldots, x_{2n}$ are arbitrary, $P(2n)$ is true.

Point 3. For every positive integer n there exists an integer $k \geq 0$ and positive integers $m_0 = 2, m_1, \ldots, m_k = n$ such that, for every $i < k$, either $m_{i+1} = 2m_i$ or $m_{i+1} = m_i - 1$. (For instance, set $m_{i+1} = 2m_i$ until $m_i \geq n$, then $m_{i+1} = m_i - 1$ until $m_i = n$.) By points 1 and 2, $P(m_0)$ is true, and $P(m_0)$ and $P(m_i)$ together imply $P(m_{i+1})$ for every $i < k$: therefore, $P(m_k)$ is true. As $m_k = n$ is arbitrary, the thesis follows.