ITT9132 Concrete Mathematics
Exercises from Week 3

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Exercise 1.8

Solve the recurrence:

$$
\begin{array}{l}
Q_0 = \alpha \ \ ; \ \ Q_1 = \beta; \\
Q_n = (1 + Q_{n-1})/Q_{n-2} \ \ , \ \ \text{for } n > 1 \ .
\end{array}
$$

Assume that $Q_n \neq 0$ for all $n \geq 0$. *Hint:* $Q_4 = (1 + \alpha)/\beta$.

Solution. Let us just start computing. We get $Q_2 = (1 + \beta)/\alpha$ and $Q_3 = (1 + ((1 + \beta)/\alpha))/\beta = (1 + \alpha + \beta)/\alpha\beta$. Then:

$$
\begin{aligned}
Q_4 &= \frac{1 + \frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\
&= \frac{\frac{\alpha\beta+1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\
&= \frac{\frac{(1+\alpha)(1+\beta)}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\
&= \frac{1 + \alpha}{\beta} \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_5 &= \frac{1+\frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} \\
&= \frac{\frac{\beta+1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} \\
&= = \alpha \;.
\end{aligned}
$$

Thus, $Q_6 = (1+\alpha)/((1+\alpha)/\beta) = \beta$, and the sequence is periodic.

Important note: Exercise 1.8 asks us to solve a *second order* recurrence with *two* initial conditions, corresponding to two consecutive indices. To be sure that the solution is a periodic sequence, we must then make sure that *two consecutive values* are repeated.

Exercise 1.16

Use the repertoire method to solve the general four-parameter recurrence

$$
\begin{aligned}
g(1) &= \alpha \;, \\
g(2n+j) &= 3g(n) + \gamma n + \beta_j \quad \text{for} \;\; j = 0, 1 \;\; \text{and} \;\; n \geq 1 \;.
\end{aligned} \tag{1}
$$

Solution. We construct a repertoire of both special solutions $g(n)$ given special values of the parameters α, β_0, β_1, γ, and special values of the parameters given special solutions: from these, the general expression for $g(n)$ is then expressed in terms of four functions $A(n)$, $B_0(n)$, $B_1(n)$, $C(n)$ as

$$
g(n) = A(n) \cdot \alpha + C(n) \cdot \gamma + B_0(n) \cdot \beta_0 + B_1(n) \cdot \beta_1 \;\; \forall n \geq 1 \;.
$$

This is possible because the system (1) is *linear* in its parameters: if $g_i(n)$ is the solution for the values $(\alpha_i, \beta_{0,i}, \beta_{1,i}, \gamma_i)$, then $\lambda_1 g_1(n) + \lambda_2 g_2(n)$ is the solution for the values

$$
(\lambda_1\alpha_1 + \lambda_2\alpha_2, \lambda_1\beta_{0,1} + \lambda_2\beta_{0,2}, \lambda_1\beta_{1,1} + \lambda_2\beta_{1,2}, \lambda_1\gamma_1 + \lambda_2\gamma_2) \,.
$$

Observe that, for $\gamma = 0$, (1) is a special case of Equation 1.17 with $\beta_1 = d = 1$. In this case, Equation 1.18 yields

$$
g_0((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \ldots \beta_{b_1} \beta_{b_0})_3 \;,
$$

which is a complete solution for $\gamma = 0$ and α, β_0, β_1 arbitrary: in particular, it links together the functions $A(n)$, $B_0(n)$ and $B_1(n)$, and yields $A(2^m+\ell) = 3^m$ for every $m \geq 0$ and $0 \leq \ell < 2^m$.

To have a complete repertoire, we consider the case $g(n) = n$ for every $n \geq 1$. Then (1) becomes:

$$\begin{aligned} 1 &= \alpha \ , \\ 2n &= 3n + \gamma n + \beta_0 \quad \forall n \geq 1 \ , \\ 2n+1 &= 3n + \gamma n + \beta_1 \quad \forall n \geq 1 \ , \end{aligned}$$

which is satisfied for $\alpha = 1$, $\gamma = -1$, $\beta_0 = 0$, $\beta_1 = 1$. This gives the relation

$$A(n) - C(n) + B_1(n) = n \ \ \forall n \geq 1 \ . \tag{2}$$

From this and 1.18 we can construct $A(n)$, $B_0(n)$, $B_1(n)$ and $C(n)$. In fact, every solution $g(n)$ of (1) is the sum of the solution $g_0(n)$ of

$$\begin{aligned} g_0(1) &= \alpha \ , \\ g_0(2n+j) &= 3g(n) + \beta_j \quad \text{for } \ j = 0, 1 \ \text{ and } \ n \geq 1 \ , \end{aligned}$$

and the solution $g_P(n)$ of

$$\begin{aligned} g_P(1) &= 0 \ , \\ g_P(2n+j) &= \gamma n \quad \text{for } \ j = 0, 1 \ \text{ and } \ n \geq 1 \ . \end{aligned}$$

If we broaden our repertoire by considering the case $g(n) = 1$ for every $n \geq 1$, (1) becomes

$$\begin{aligned} 1 &= \alpha \ , \\ 1 &= 3 + \gamma n + \beta_0 \quad \forall n \geq 1 \ , \\ 1 &= 3 + \gamma n + \beta_1 \quad \forall n \geq 1 \ , \end{aligned}$$

which is satisfied for $\alpha = 1$, $\gamma = 0$, $\beta_0 = \beta_1 = -2$: this gives the relation

$$A(n) - 2B_0(n) - 2B_1(n) = 1 \ \ \forall n \geq 1 \ ,$$

which allows to express $B_0(n)$ in terms of the simpler functions $A(n)$ and $D(n) = A(n) + B_1(n)$.

Exercise 2.2

Simplify the expression $x \cdot ([x > 0] - [x < 0])$.

Solution. If $x > 0$ then the expression has value $x \cdot (1 - 0) = x$. If $x = 0$ then the expression has value $0 \cdot (0 - 0) = 0$. If $x < 0$ then the expression has value $x \cdot (0 - 1) = -x$. Thus, $x \cdot ([x > 0] - [x < 0]) = |x|$

.

Exercise 2.13

Use the repertoire method to find a closed form for $\sum_{k=0}^{n}(-1)^k k^2$.

Solution. The function $g(n) = \sum_{k=0}^{n}(-1)^k k^2$ is a special solution of the recurrence equation:

$$\begin{aligned} R_0 &= \alpha\,, \\ R_n &= R_{n-1} + (-1)^n(\beta + \gamma n + \delta n^2) \quad \text{for} \;\; n \geq 1 \end{aligned}$$

for the special values $\alpha = \beta = \gamma = 0$, $\delta = 1$. As we know that we can express

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

for special functions $A(n)$, $B(n)$, $C(n)$ and $D(n)$, if we manage to find $D(n)$ in closed form, then that will be the closed form of $g(n)$.

Let us use the repertoire method. First of all, for $\alpha = 1$, $\beta = \gamma = \delta = 0$ we find $A(n) = 1$ for every $n \geq 0$. The next step should *not* be to put $R_n = 1$ for every $n \geq 0$, as we already know that this is associate to the special values $\alpha = 1$, $\beta = \gamma = \delta = 0$. Instead, we put $R_n = (-1)^n$, which corresponds to $\alpha = 1$, $\beta = 2$, $\gamma = \delta = 0$ and yields $A(n) + 2B(n) = (-1)^n$: as we know that $A(n) = 1$ for every $n \geq 0$, this means $2B(n) = (-1)^n - 1$ and thus

$$B(n) = ((-1)^n - 1)/2 = -\left[n \text{ is odd}\right]$$

. The third step will be to put $R_n = (-1)^n \cdot n$. This corresponds to the recurrence:

$$\begin{aligned} 0 &= \alpha\,, \\ (-1)^n n &= (-1)^{n-1}(n-1) + (-1)^n(\beta + \gamma n + \delta n^2) \\ &= (-1)^n(1-n) + (-1)^n(\beta + \gamma n + \delta n^2) \\ &= (-1)^n \cdot \big((\beta+1) + (\gamma-1)n + \delta n^2\big) \quad \forall n \geq 1\,, \end{aligned}$$

which is satisfied if and only if $\alpha = \delta = 0$, $\beta = -1$, and $\gamma = 2$. We thus get the equation:

$$-B(n) + 2C(n) = (-1)^n n\,.$$

The fourth step will be to put $R_n = (-1)^n n^2$. This corresponds to the recurrence:

$$\begin{array}{rcl}
0 &=& \alpha\,,\\
(-1)^n n^2 &=& (-1)^{n-1}(n-1)^2 + (-1)^n(\beta+\gamma n+\delta n^2)\\
&=& (-1)^{n-1}(n^2-2n+1) + (-1)^n(\beta+\gamma n+\delta n^2)\\
&=& (-1)^n(-n^2+2n-1) + (-1)^n(\beta+\gamma n+\delta n^2)\\
&=& (-1)^n \cdot \big((\beta-1)+(\gamma+2)n+(\delta-1)n^2\big) \quad \forall n \geq 1\,,
\end{array}$$

which is satisfied if and only if $\beta = 1$, $\gamma = -2$, and $\delta = 2$. We thus get:

$$B(n) - 2C(n) + D(n) = (-1)^n n^2\,.$$

At this point, we have a full system of equations:

$$\begin{array}{lllll}
A(n) & & & & = 1\\
A(n) & +2B(n) & & & = (-1)^n\\
 & -B(n) & +2C(n) & & = (-1)^n n\\
 & B(n) & -2C(n) & +2D(n) & = (-1)^n n^2
\end{array}$$

from which we want to find $D(n)$. But by adding together the third and fourth equation we immediately find $2D(n) = (-1)^n \cdot (n + n^2)$ Then $g(n) = D(n) = (-1)^n(n^2 + n)/2 = (-1)^n S_n$.

Exercise 2.21 (part 1)

Evaluate the sum $S_n = \sum_{k=0}^{n}(-1)^{n-k}$ by the perturbation method, assuming that $n \geq 0$.

Solution. On the one hand,

$$\begin{array}{rcl}
S_{n+1} &=& \displaystyle\sum_{0\leq k\leq n+1}(-1)^{n+1-k}\\
&=& \displaystyle\sum_{0\leq k\leq n}(-1)^{n+1-k} + 1\\
&=& -S_n + 1\,;
\end{array}$$

next,

$$\begin{array}{rcl}
S_{n+1} &=& (-1)^{n+1} + \displaystyle\sum_{1\leq k\leq n+1}(-1)^{n+1-k}\\
&=& (-1)^{n+1} + \displaystyle\sum_{0\leq k\leq n}(-1)^{n-k}\\
&=& (-1)^{n+1} + S_n\,.
\end{array}$$

Together, the two equalities above yield $2S_n = 1 - (-1)^{n+1} = 1 + (-1)^n$, so that:

$$S_n = \frac{1 + (-1)^n}{2} = [n \text{ is even}] \ .$$