ITT9132 Concrete Mathematics
Exercises from Week 4

Silvio Capobianco

Last update: 20 February 2019

A note on the repertoire method

Consider a recurrence equation of the form:

$$\begin{aligned} g(0) &= \alpha \,, \\ g(n+1) &= \Phi(g(n)) + \Psi(n; \beta, \gamma, \ldots) \quad \text{for} \;\; n \geq 0 \end{aligned} \tag{1}$$

where:

1. Φ is linear *in* g, *i.e.*, if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.

   No hypotheses are made on the dependence of g on n.

2. Ψ is a linear function of the $m-1$ parameters $\beta, \gamma, \ldots$

   No hypotheses are made on the dependence of Ψ on n.

Let a a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfy the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$

2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then functions $A(n), B(n), C(n), \ldots$, one per parameter, are uniquely determined such that, however given $\alpha, \beta, \gamma, \ldots$, the solution of the recurrence equation (1) is:

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \ldots$$

Exercise A.1

Use the repertoire method to solve the following general recurrence:

$$\begin{aligned} g(0) &= \alpha \,, \\ g(n+1) &= 2g(n) + \beta n + \gamma \quad \text{for } n \geq 0 \,. \end{aligned} \tag{2}$$

Solution. The recurrence (2) has the form (1) with $\Phi(g) = 2g$ and $\Psi(n; \beta, \gamma) = \beta n + \gamma$, which are linear in g and in β and γ, respectively: therefore we can apply the repertoire method. The special case $g(n) = 1$ for every $n \geq 0$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, -1)$: thus,

$$A(n) - C(n) = 1 \,.$$

The special case $g(n) = n$ for every $n \geq 0$ corresponds to $(\alpha, \beta, \gamma) = (0, -1, 1)$: thus,

$$-B(n) + C(n) = n \,.$$

The special case $g(n) = 2^n$ for every $n \geq 0$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 0)$: thus,

$$A(n) = 2^n \ \text{ and consequently}\,, \ C(n) = 2^n - 1 \ \text{ and } \ B(n) = 2^n - 1 - n \,.$$

The general solution of (2) is then:

$$\begin{aligned} g(n) &= \alpha \cdot 2^n + \beta \cdot (2^n - 1 - n) + \gamma \cdot (2^n - 1) \\ &= (\alpha + \beta + \gamma) \cdot 2^n - \beta n - (\beta + \gamma) \,. \end{aligned}$$

Exercise A.2

What if the recurrence (2) had been

$$\begin{aligned} g(0) &= \alpha \,, \\ g(n+1) &= \delta g(n) + \beta n + \gamma \quad \text{for } n \geq 0 \,. \end{aligned} \tag{3}$$

instead?

Solution. The recurrence (3), considered as a family of recurrence equations parameterized by $(\alpha, \beta, \gamma, \delta)$, does *not* have the form (1)! Here, the function Φ depends on both the function g and the parameter δ: because of this, in general $g_1(n) + g_2(n)$ is not the solution for $(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2)$.

However, for every *fixed* δ, (3) *does* have the form (1) with $\Phi(g) = \delta g$ and $\Psi(n; \beta, \gamma) = \beta n + \gamma$: thus, for every fixed δ, we can use the repertoire method to find three functions $A_\delta(n), B_\delta(n), C_\delta(n)$ such that

$$g_\delta(n) = \alpha \cdot A_\delta(n) + \beta \cdot B_\delta(n) + \gamma \cdot C_\delta(n)$$

for every $n \geq 0$. By reasoning as before, the choice $g_\delta(n) = 1$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 1 - \delta)$, thus

$$A_\delta(n) + (1 - \delta)C(n) = 1 \ : \qquad (4)$$

the factor $1 - \delta$ in front of $C_\delta(n)$ rings a bell, and suggests we might have to be careful about the cases $\delta = 1$ and $\delta \neq 1$. Choosing $g_\delta(n) = n$ corresponds to $(\alpha, \beta, \gamma) = (0, 1 - \delta, 1)$, thus

$$(1 - \delta)B_\delta(n) + C_\delta(n) = n \ . \qquad (5)$$

We are left with one triple of values to choose. As we had put $g(n) = 2^n$ when $\delta = 2$, we are tempted to just put $g(n) = \delta^n$: but if $\delta = 1$ this would be the same as $g(n) = 1$, which we have already considered. We will then deal separately with the cases $\delta = 1$ and $\delta \neq 1$.

Let us start with the latter. For $\delta \neq 1$ the choice $g_\delta(n) = \delta^n$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 0)$, thus

$$A_\delta(n) = \delta^n \ : \qquad (6)$$

by combining this with (4) and (5) we find

$$C_\delta(n) = \frac{1 - A_\delta(n)}{1 - \delta} = \frac{1 - \delta^n}{1 - \delta} = 1 + \delta + \ldots + \delta^{n-1}$$

and

$$B(n) = \frac{n - C_\delta(n)}{1 - \delta} = \frac{n - 1 - \delta - \ldots - \delta^{n-1}}{1 - \delta} \ .$$

Let us now consider the case $\delta = 1$. Then (4) becomes $A_1(n) = 1$ and (5) becomes $C_1(n) = n$: for the last case, we set $g_1(n) = n^2$, which corresponds to $(\alpha, \beta, \gamma) = (0, 2, 1)$, and find

$$2B_1(n) + C_1(n) = n^2 \ , \qquad (7)$$

which yields $B_1(n) = (n^2 - n)/2$.

Exercise 2.6

What is the value of $\sum_k [1 \le j \le k \le n]$ as a function of j and n?

Solution. If $j < 1$ or $j > n$, then the sum is empty and its value is zero. If $1 \le j \le n$, then the sum has $n - j + 1$ nonzero summands, each having value 1. Therefore, $\sum_k [1 \le j \le k \le n] = (n - j + 1) \cdot [1 \le j \le n]$.

Exercise 2.14

Use multiple sums to evaluate

$$\sum_{k=1}^{n} k \cdot 2^k$$

Solution. Write $k = \sum_{j=1}^{k} 1$. Then:

$$\begin{aligned}
\sum_{k=1}^{n} k \cdot 2^k &= \sum_{k=1}^{n} \left(\sum_{j=1}^{k} 1 \right) \cdot 2^k \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k} 1 \cdot 2^k \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} 2^k
\end{aligned}$$

Clearly,

$$\begin{aligned}
\sum_{k=j}^{n} 2^k &= 2^j \cdot \sum_{k=0}^{n-j} 2^k \\
&= 2^j \cdot \left(2^{n-j+1} - 1 \right) \\
&= 2^{n+1} - 2^j
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{n} k \cdot 2^k &= \sum_{j=1}^{n} \left(2^{n+1} - 2^j\right) \\
&= \sum_{j=1}^{n} 2^{n+1} - \sum_{j=1}^{n} 2^j \\
&= n \cdot 2^{n+1} - 2 \cdot \sum_{j=0}^{n-1} 2^j \\
&= n \cdot 2^{n+1} - 2 \cdot (2^n - 1) \\
&= n \cdot 2^{n+1} - 2^{n+1} + 2 \\
&= (n-1) \cdot 2^{n+1} + 2
\end{aligned}$$

Exercise 2.15

Evaluate $\text{⧉}_n = \sum_{k=1}^{n} k^3$ by the text's Method 5 as follows: First write $\text{⧉}_n + \square_n = 2\sum_{1 \le j \le k \le n} jk$; then apply (2.33).

Solution. Recall that $\square_n = \sum_{k=1}^{n} k^2$. Then:

$$\begin{aligned}
\text{⧉}_n + \square_n &= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \\
&= \sum_{k=1}^{n} k^2(k+1) \\
&= 2\sum_{k=1}^{n} k \cdot \frac{k(k+1)}{2} \\
&= 2\sum_{k=1}^{n} k \cdot \sum_{j=1}^{k} j \\
&= 2\sum_{1 \le j \le k \le n} jk \;.
\end{aligned}$$

By (2.33), whatever the summands a_k are,

$$\sum_{1 \le j \le k \le n} a_j a_k = \frac{1}{2}\left(\sum_{k=1}^{n} a_k^2 + \left(\sum_{k=1}^{n} a_k\right)^2\right) \;:$$

in our case, $a_k = k$, and

$$\Cube_n + \Box_n = \sum_{k=1}^{n} k^2 + \left(\sum_{k=1}^{n} k\right)^2 = \Box_n + \left(\sum_{k=1}^{n} k\right)^2 \; ,$$

which yields $\Cube_n = S_n^2$.

Exercise 2.21

Evaluate the sums $S_n = \sum_{k=0}^{n}(-1)^{n-k}$, $T_n = \sum_{k=0}^{n}(-1)^{n-k}k$, and $U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$ by the perturbation method, assuming that $n \geq 0$.

Solution. By applying the permutation $p(k) = n - k$ we see that $S_n = [n \text{ is even}]$. Let's try to reach the same result via the perturbation method. First,

$$\begin{aligned} S_{n+1} &= \sum_{0 \leq k \leq n+1} (-1)^{n+1-k} \\ &= \sum_{0 \leq k \leq n} (-1)^{n+1-k} + 1 \\ &= -S_n + 1 \; ; \end{aligned}$$

next,

$$\begin{aligned} S_{n+1} &= (-1)^{n+1} + \sum_{1 \leq k \leq n+1} (-1)^{n+1-k} \\ &= (-1)^{n+1} + \sum_{0 \leq k \leq n} (-1)^{n-k} \\ &= (-1)^{n+1} + S_n \; . \end{aligned}$$

Together, the two equalities above yield $2S_n = 1 - (-1)^{n+1} = 1 + (-1)^n$, so that:

$$S_n = \frac{1 + (-1)^n}{2} = [n \text{ is even}] \; .$$

For T_n we use a similar trick. First,

$$\begin{aligned} T_{n+1} &= \sum_{0 \leq k \leq n} (-1)^{n+1-k} k + n + 1 \\ &= -T_n + n + 1 \; ; \end{aligned}$$

next,

$$
\begin{aligned}
T_{n+1} &= 0 + \sum_{1 \le k \le n+1} (-1)^{n+1-k} k \\
&= \sum_{0 \le k \le n} (-1)^{n-k}(k+1) \\
&= T_n + S_n \ ;
\end{aligned}
$$

together these yield $2T_n = n + 1 - S_n$. But as $S_n = [n \text{ is even}]$, $1 - S_n = [n \text{ is odd}]$: thus,

$$T_n = \frac{n + [n \text{ is odd}]}{2} \ .$$

With U_n the trick will be similar as with T_n, but we will have to be careful about the square:

$$
\begin{aligned}
-U_n + (n+1)^2 &= \sum_{0 \le k \le n} (-1)^{n-k}(k+1)^2 \\
&= \sum_{0 \le k \le n} (-1)^{n-k}(k^2 + 2k + 1) \\
&= U_n + 2T_n + S_n \ ,
\end{aligned}
$$

which yields $2U_n = (n+1)^2 - 2T_n - S_n$. But

$$2T_n + S_n = n + [n \text{ is odd}] + [n \text{ is even}] = n + 1 \ :$$

thus, $U_n = (n^2 + n)/2$.