ITT9132 Concrete Mathematics
Exercises from Week 5

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Exercise 2.8

What is the value of $0^{\underline{m}}$, when m is a given integer?

Solution. Let us consider the three cases $m > 0$, $m = 0$, $m < 0$.

- If $m > 0$, then $0^{\underline{m}} = 0 \cdot (-1) \cdots (-m+1) = 0$, since the factor 0 is surely present.
- If $m = 0$ then $0^{\underline{m}} = 1$ as an empty product.
- If $m < 0$ then we should have $1 = 0^{\underline{0}} = 0^{\underline{m}} \cdot (0-m)^{\underline{-m}}$. For $n > 0$ ve have $n^{\underline{n}} = n!$. Thus, $0^{\underline{m}} = 1/(-m)! = 1/|m|!$.

Summarizing: $0^{\underline{m}} = \dfrac{1}{|m|!} \cdot [m \leqslant 0]$.

Exercise 2.10

The text derives the following formula for the difference of a product:

$$\Delta(uv) = u\Delta v + Ev\Delta u \; . \tag{1}$$

How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not?

Solution. The equation above is obtained as follows:

$$\begin{aligned}
\Delta(uv)(x) &= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\
&= \Delta u(x) Ev(x) + u(x)\Delta v(x)
\end{aligned}$$

But the following derivation is also licit:

$$\begin{aligned}\Delta(uv)(x) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x)\\ &= Eu(x)\Delta v(x) + \Delta u(x)v(x)\end{aligned}$$

So there actually is a symmetry: $u\Delta v + Ev\Delta u = v\Delta u + Eu\Delta v$.

Exercise 2.16

Prove that $x^{\underline{m}}/(x-n)^{\underline{m}} = x^{\underline{n}}/(x-m)^{\underline{n}}$ unless one of the denominators is zero.

Solution. If b and d are not zero, then $a/b = c/d$ if and only if $a \cdot d = b \cdot c$. But if x is neither n nor m, then $x^{\underline{m}} \cdot (x-m)^{\underline{n}}$ and $x^{\underline{n}} \cdot (x-n)^{\underline{m}}$ are both equal to $x^{\underline{n+m}}$.

Exercise 2.17

Show that the following formulas can be used to convert between rising and falling factorial powers, for all integers m:

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \frac{1}{(x-1)^{\underline{-m}}}\,;$$

$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{\overline{-m}}}\,.$$

(The answer to exercise 2.9 defines $x^{\overline{-m}}$.)

Solution. The equalities $x^{\overline{m}} = (x+m-1)^{\underline{m}}$ and $x^{\underline{m}} = (x-m+1)^{\overline{m}}$ follow immediately from the definition. Next,

$$x^{\overline{m}} = x\cdot(x+1)\cdots(x+m-1) = (-1)^m\cdot(-x)\cdot(-x-1)\cdots(-x-m+1) = (-1)^m(-x)^{\underline{m}}$$

and analogous for $x^{\overline{m}} = (-1)^m(-x)^{\underline{m}}$. Finally,

$$(x-1)^{\underline{-m}} = \frac{1}{x(x+1)\cdots(x+m-1)} = \frac{1}{x^{\overline{m}}}\,,$$

and similarly,

$$(x+1)^{\overline{-m}} = \frac{1}{x(x-1)\cdots(x-m+1)} = \frac{1}{x^{\underline{m}}}\,.$$

Exercise 2.23

Evaluate $\sum_{k=1}^{n}(2k+1)/k(k+1)$ in two ways:

1. Replace $1/k(k+1)$ by the "partial fractions" $1/k - 1/(k+1)$.
2. Sum by parts.

Solution. By method 1 we get:

$$\begin{aligned}\sum_{k=1}^{n}\frac{2k+1}{k(k+1)} &= 2\sum_{k=1}^{n}\frac{1}{k+1}+\sum_{k=1}^{n}\frac{1}{k(k+1)} \\ &= 2\cdot(H_{n+1}-1)+\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) \\ &= 2H_{n+1}-2+1-\frac{1}{n+1} \\ &= 2H_n-\frac{n}{n+1}\ ,\end{aligned}$$

as $H_{n+1} = H_n + 1/(n+1)$.

To use method 2, we need to express $(2k+1)/k(k+1)$ as $u\Delta v$ for suitable u and v. If we choose $u(x) = 2x+1$ and $\Delta v(x) = 1/x(x+1) = (x-1)^{\underline{-2}}$, then $\Delta u(x) = 2$ and $v(x) = -(x-1)^{\underline{-1}} = -1/x$, thus:

$$\begin{aligned}\sum_{k=1}^{n}\frac{2k+1}{k(k+1)} &= \sum_{1}^{n+1}u(x)\Delta v(x)\delta x \\ &= u(x)v(x)\Big|_{x=1}^{x=n+1}-\sum_{1}^{n+1}Ev(x)\Delta u(x)\delta x \\ &= -\frac{2x+1}{x}\Big|_{x=1}^{x=n+1}+\sum_{1}^{n+1}\frac{2}{x+1}\delta x\ .\end{aligned}$$

The first summand is simply $1-1/(n+1)$. For the second, we know that $\sum_{1}^{n+1}\Delta g(x)\delta x = g(n+1)-g(1)$: for $\Delta g(x) = 1/(x+1)$ it is clearly $g(x) = H_x$, thus

$$\sum_{1}^{n+1}\frac{2}{x+1} = 2(H_{n+1}-H_1) = 2H_n+\frac{2}{n+1}-2\ .$$

Putting everything together, $\sum_{k=1}^{n}(2k+1)/k(k+1) = 2H_n - n/(n+1)$, as we had previously found.

Exercise 2.28

At what point does the following derivation go astray?

$$
\begin{aligned}
1 &= \sum_{k\geqslant 1} \frac{1}{k\cdot(k+1)} && (2)\\
&= \sum_{k\geqslant 1}\left(\frac{k}{k+1} - \frac{k-1}{k}\right) && (3)\\
&= \sum_{k\geqslant 1}\sum_{j\geqslant 1}\left(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]\right) && (4)\\
&= \sum_{j\geqslant 1}\sum_{k\geqslant 1}\left(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]\right) && (5)\\
&= \sum_{j\geqslant 1}\sum_{k\geqslant 1}\left(\frac{k}{j}[k=j-1] - \frac{j}{k}[k=j+1]\right) && (6)\\
&= \sum_{j\geqslant 1}\left(\frac{j-1}{j} - \frac{j}{j+1}\right) && (7)\\
&= \sum_{j\geqslant 1} \frac{-1}{j\cdot(j+1)} && (8)\\
&= -1 && (9)
\end{aligned}
$$

Solution. Let us check each passage in detail.

- (2) is correct. The right hand side is the limit for $n \to \infty$ of the telescopic sum $\sum_{1\leqslant k\leqslant n}(1/k - 1/(k+1)) = 1 - 1/(n+1)$.
- (3) is correct. We are rewriting the generic summand into a different form, whose value is the same as the previous.
- (4) is correct. We are replacing each summand with a new sum whose summands are all zero except finitely many.
- (5) is **WRONG!** We are re-arranging the terms of a sum which is not absolutely convergent: this operation is not guaranteed to preserve the value of the sum.
- (6) is correct. We are rewriting the conditions in Iverson brackets without altering them.

- (7) is correct. We are summing over k, which gives an immediate result since for every j only finitely many summands are nonzero.
- (8) is correct.
- (9) is correct.