ITT9132 Concrete Mathematics
Exercises from Week 6

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Exercise 3.2

Give an explicit formula for the integer nearest to the real number x. Do this in the two cases when an integer plus 1/2 is rounded up or down.

Solution. Put $x = n + t$ with n integer and $0 \leqslant t < 1$. Rounding x to the nearest integer must yield $n = \lfloor x \rfloor$ when $t < 1/2$, and $n + 1 = \lceil x \rceil$ when $t > 1/2$.

This can be done by rounding x to $\lfloor x + 1/2 \rfloor$. In fact, $\lfloor x + 1/2 \rfloor = \lfloor n + 1/2 + t \rfloor$ is n if $t < 1/2$, and $n + 1$ if $t > 1/2$. We also observe that $\lfloor x + 1/2 \rfloor = n + 1$ if $t = 1/2$, *i.e.*, this is the choice that corresponds to rounding *up*.

Another option is to reason as follows: Being $x = \lfloor x \rfloor + \{x\}$, rounding up means turning x into $\lfloor x \rfloor$ if $\{x\} < 1/2$, and into $\lceil x \rceil = \lfloor x \rfloor + 1$ if $\{x\} \geqslant 1/2$. Then we can just use Iverson's brackets and round x to $\lfloor x \rfloor + [\{x\} \geqslant 1/2]$.

Are there any options for rounding *down*? We may try reasoning "by symmetry" and swapping floor with ceiling, plus with minus: that is, round x to $\lceil x - 1/2 \rceil = \lceil n - 1/2 + t \rceil$. And in fact, we immediately check that this quantity is n for $t < 1/2$ and $n + 1$ for $t > 1/2$. What about $t = 1/2$? We quickly get $\lceil (n + 1/2) - 1/2 \rceil = \lceil n \rceil = n$. So this is the function that rounds down, as required.

Exercise 3.3

Let m and n be positive integers and let α be an irrational number greater than n. Evaluate $\lfloor \lfloor m\alpha \rfloor n/\alpha \rfloor$.

Solution. The floor inside the floor is threatening trouble, so we should try to make it disappear. Write $m\alpha = \lfloor m\alpha \rfloor + \{m\alpha\}$. Then:

$$\frac{\lfloor m\alpha \rfloor n}{\alpha} = \frac{(m\alpha - \{m\alpha\})n}{\alpha} = mn - \frac{\{m\alpha\} n}{\alpha} .$$

By hypothesis, $1 \leqslant n < \alpha$. Moreover, α is irrational, so $m\alpha$ is not an integer and $\{m\alpha\}$ is positive. Consequently, $0 < \{m\alpha\} \cdot (n/\alpha) < 1 \cdot 1$. We can thus conclude that:

$$\begin{aligned} \left\lfloor \frac{\lfloor m\alpha \rfloor n}{\alpha} \right\rfloor &= \left\lfloor mn - \frac{\{m\alpha\} n}{\alpha} \right\rfloor \\ &= mn + \left\lfloor \frac{-\{m\alpha\} n}{\alpha} \right\rfloor \\ &= mn - \left\lceil \frac{\{m\alpha\} n}{\alpha} \right\rceil \\ &= mn - 1 . \end{aligned}$$

Note that we used the rule $\lfloor n + x \rfloor = n + \lfloor x \rfloor$, which holds whatever integer n and real x are. To apply it correctly, we must keep the "plus" sign outside the floor and not change x. This means that $\lfloor n - x \rfloor$ is $n + \lfloor -x \rfloor = n - \lceil x \rceil$, and not (in general) $n - \lfloor x \rfloor$.

Exercise 3.6

Can something interesting be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous, monotonically *decreasing* function that takes integer values only when x is an integer?

Solution. If $f(x)$ is continuous and strictly decreasing and only takes integer values on integer numbers, then $g(x) = -f(x)$ is continuous and strictly *in*creasing and only takes integer values on integer numbers. Then:

$$\lfloor f(x) \rfloor = -\lceil g(x) \rceil = -\lceil g(\lceil x \rceil) \rceil = \lfloor f(\lceil x \rceil) \rfloor ,$$

and similarly, $\lceil f(x) \rceil = \lceil f(\lfloor x \rfloor) \rceil$.

Exercise 3.10

Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor \tag{1}$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

Solution. We observe that

$$\begin{aligned}\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor &= \left\lceil \frac{2x+1}{2} \right\rceil - \left(\left\lceil \frac{2x+1}{4} \right\rceil - \left\lfloor \frac{2x+1}{4} \right\rfloor \right) \\ &= \left\lceil x + \frac{1}{2} \right\rceil - \left[\frac{2x+1}{4} \text{ is not an integer} \right] .\end{aligned}$$

(Do not forget that x is a real number.) But $(2x+1)/4 = k$ is an integer if and only if $x = (4k-1)/2 = 2k - 1/2$: in this case, $\lceil x + 1/2 \rceil = 2k = \lceil x \rceil$. Otherwise, we know that

$$\left\lceil x + \frac{1}{2} \right\rceil - 1 = \left\lceil (x+1) - \frac{1}{2} \right\rceil - 1 = \left\lceil x - \frac{1}{2} \right\rceil$$

is $\lfloor x \rfloor$ if $\{x\} < 1/2$, and $\lceil x \rceil$ if $\{x\} \geqslant 1/2$.

Exercise 3.12

Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \tag{2}$$

for all integers n and all positive integers m. (This identity gives us another way to convert ceilings to floors and vice versa, instead of using the reflective law (3.4).)

Solution. The closed interval $[n/m..(n+m-1)/m]$ has size $1 - 1/m$, and can thus contain at most one integer: in this case, such integer must coincide with both $\lceil n/m \rceil$ and $\lfloor (n+m-1)/m \rfloor$. However, of the m consecutive integers $n, n+1, \ldots, n+m-1$, exactly one is divisible by m: if x is this number, then $x/m \in [n/m, (n+m-1)/m]$ is the common value of $\lceil n/m \rceil$ and $\lfloor (n+m-1)/m \rfloor$.

Exercise 3.13

Let α and β be positive reals. Consider the following statements:

1. $\mathrm{Spec}(\alpha)$ and $\mathrm{Spec}(\beta)$ partition the positive integers, *i.e.*, every positive integer n belongs to exactly one between $\mathrm{Spec}(\alpha)$ and $\mathrm{Spec}(\beta)$.

2. α and β are irrational and $1/\alpha + 1/\beta = 1$.

Prove that statement 2 implies statement 1.

Solution. We recall that, for a positive real x, the number $N(x, n)$ of elements in $\mathrm{Spec}(x)$ not greater than n satisfies

$$N(x, n) = \left\lceil \frac{n+1}{x} \right\rceil - 1$$

Suppose that point 2 is satisfied. Then α and β, being irrational, must be different (otherwise $\alpha = \beta = 2$). Also, $(n+1)/\alpha$ is not an integer (because α is irrational) and:

$$N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1 = \left\lfloor \frac{n+1}{\alpha} \right\rfloor = \frac{n+1}{\alpha} - \left\{ \frac{n+1}{\alpha} \right\},$$

and similarly for $(n+1)/\beta$. Hence,

$$N(\alpha, n) + N(\beta, n) = \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) (n+1) - \left(\left\{ \frac{n+1}{\alpha} \right\} + \left\{ \frac{n+1}{\beta} \right\} \right)$$

By hypothesis, $1/\alpha + 1/\beta = 1$. Then the rightmost term in open parentheses is the sum of the fractional parts of two non-integer numbers whose sum is an integer, and is therefore equal to 1. Therefore, $N(\alpha, n) + N(\beta, n) = n + 1 - 1 = n$ for every positive integer n: then also, for every n, either $N(\alpha, n+1) = N(\alpha, n)+1$ and $N(\beta, n+1) = N(\beta, n)$, or $N(\alpha, n+1) = N(\alpha, n)$ and $N(\beta, n+1) = N(\beta, n) + 1$, that is, each integer larger than 1 goes into exactly one of the two spectra. As $1/\alpha + 1/\beta = 1$ and $\alpha \neq \beta$, one of them is smaller than 2 and the other is greater, and $n = 1$ goes into the spectrum of the former: this allows us to conclude that $\mathrm{Spec}(\alpha)$ and $\mathrm{Spec}(\beta)$ partition the positive integers.

Exercise C.2

Prove equation (3.24): for every integer n and positive integer m,

$$\left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + \ldots + \left\lceil \frac{n-m+1}{m} \right\rceil = n\,.$$

Use the result to prove (3.25).

Solution. Write $n = qm + r$ with $q, r \in \mathbb{Z}$ and $0 \leqslant r < m$. Then for every k from 1 to m:

$$\left\lceil \frac{n-k+1}{m} \right\rceil = q + \left\lceil \frac{r-k+1}{m} \right\rceil .$$

Now, for k between 0 and $m-1$, $\left\lceil \frac{r-k+1}{m} \right\rceil$ is 1 if $r-k+1 > 0$ (that is, $k \leqslant r$) and 0 otherwise. Then:

$$\begin{aligned} \sum_{k=1}^{m} \left\lceil \frac{n-k+1}{m} \right\rceil &= \sum_{k=1}^{m} \left(q + \left\lceil \frac{r-k+1}{m} \right\rceil \right) \\ &= qm + \sum_{k=1}^{m} [k \leqslant r] \\ &= qm + r \\ &= n \,. \end{aligned}$$

Now, (3.24) holds for *every* integer n and positive integer m. If we want to prove (3.25), we can just exploit it:

$$\begin{aligned} \sum_{k=1}^{m} \left\lfloor \frac{n+k-1}{m} \right\rfloor &= -\sum_{k=1}^{m} \left\lceil -\frac{n+k-1}{m} \right\rceil \\ &= -\sum_{k=1}^{m} \left\lceil \frac{-n-k+1}{m} \right\rceil \\ &= -(-n) \ \text{ by (3.24)} \\ &= n \,. \end{aligned}$$