ITT9132 Concrete Mathematics
Exercises from Week 9

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Exercise 1

Solve the recurrence:

$$\begin{aligned} T_0 &= 1\,; \\ T_n &= 2T_{n-1} + \left(\frac{3}{2}\right)^n + 2^n H_n \quad \forall n \geqslant 1\,. \end{aligned} \tag{1}$$

Solution. The system (1) has the form

$$\begin{aligned} a_0 T_0 &= 1\,; \\ a_n T_n &= b_n T_{n-1} + c_n \quad \forall n \geqslant 1 \end{aligned}$$

with

$$a_n = 1\ ;\ b_n = 2\ ;\ c_n = \left(\frac{3}{2}\right)^n + H_n\,.$$

This suggests using a summation factor:

$$s_0 = 1\ ;\ s_n = \prod_{j=1}^{n} \frac{a_{j-1}}{b_j} = \frac{1}{2^n} \quad \forall n \geqslant 1\,.$$

Then, by putting $U_n = s_n a_n T_n = T_n / 2^n$ and simplifying, we get

$$\begin{aligned} U_0 &= 1\,; \\ U_n &= U_{n-1} + \left(\frac{3}{4}\right)^n + H_n \quad \forall n \geqslant 1\ : \end{aligned}$$

which clearly has the solution

$$\begin{aligned} U_n &= 1 + \sum_{k=1}^{n} \left(\left(\frac{3}{4} \right)^k + H_k \right) \\ &= \sum_{k=0}^{n} \left(\frac{3}{4} \right)^k + \sum_{k=1}^{n} H_k \\ &= \frac{4^{n+1} - 3^{n+1}}{4^n} + (n+1)H_n - n \end{aligned}$$

In the end, the solution to (1) is:

$$T_n = \frac{4^{n+1} - 3^{n+1}}{2^n} + 2^n \cdot ((n+1)H_n - n) \ .$$

Exercise 2

Solve the recurrence:

$$\begin{aligned} T_0 &= 1 ; \\ nT_n &= 2T_{n-1} + \frac{2^n}{n!} \left(1 + \frac{n}{3^n} \right) \quad \forall n \geqslant 1 . \end{aligned} \tag{2}$$

Solution. Equation (2) has the form

$$a_n T_n = b_n T_{n-1} + c_n$$

with:

$$a_0 = 1 ; \quad a_n = n \text{ for every } n \geqslant 1 ; \quad b_n = 2 ; \quad c_n = \frac{2^n}{n!} \left(1 + \frac{n}{3^n} \right) .$$

This suggests using a summation factor s_n such that $s_n b_n = s_{n-1} a_{n-1}$ for every $n \geqslant 1$. We must be a bit careful, because $a_n = n$ only for $n \geqslant 1$, while $a_0 = 1$; so we have to determine separately not only s_0, but also s_1. We have:

$$s_0 = 1 ; \quad s_1 = \frac{a_0}{b_1} = \frac{1}{2} ; \quad s_n = s_{n-1} \cdot \frac{a_{n-1}}{b_n} = s_{n-1} \cdot \frac{n-1}{2} \quad \text{for every } n \geqslant 2 .$$

The last recurrence has the solution $s_n = (n-1)!/2^n$, which also holds for $n = 1$ as $0! = 1$. By multiplying (2) by s_n and putting

$$U_n = s_n a_n T_n = \frac{(n-1)!}{2^n} n T_n = \frac{n!}{2^n} T_n$$

we get:

$$\begin{aligned}
u_0 &= 1\,; \\
U_n &= U_{n-1} + \frac{(n-1)!}{2^n} \cdot \frac{2^n}{n!} \left(1 + \frac{n}{3^n}\right) \\
&= U_{n-1} + \frac{1}{n} \left(1 + \frac{n}{3^n}\right) \\
&= U_{n-1} + \frac{1}{n} + \frac{1}{3^n}
\end{aligned}$$

which has the solution:

$$\begin{aligned}
U_n &= 1 + \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{3^k}\right) \\
&= \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=0}^{n} \frac{1}{3^k} \\
&= H_n + \frac{1 - (1/3)^{n+1}}{1 - 1/3} \\
&= H_n + \frac{1}{2} \cdot \left(3 - \frac{1}{3^n}\right) .
\end{aligned}$$

Then the solution of our original recurrence is:

$$T_n = \frac{2^n}{n!} U_n = \frac{2^n}{n!} H_n + \frac{2^{n-1}}{n!} \cdot \left(3 - \frac{1}{3^n}\right) .$$

Exercise 3

Express $\sum_{1 \leqslant k \leqslant n} k \cdot 2^{-k}$ as a function of n, and evaluate $\sum_{k \geqslant 1} k \cdot 2^{-k}$.

Solution. We can compute $\sum_{1 \leqslant k \leqslant n} k \cdot 2^{-k}$ in two different ways:

- Perturbation method:

Let $S_n = \sum_{1 \leqslant k \leqslant n} k \cdot 2^{-k}$: then

$$\begin{aligned} S_n + (n+1) \cdot 2^{-n-1} &= \frac{1}{2} + \sum_{k=2}^{n+1} k \cdot 2^{-k} \\ &= \frac{1}{2} + \sum_{k=1}^{n} (k+1) \cdot 2^{-k-1} \\ &= \frac{1}{2} + \frac{1}{2} \left(\sum_{k=1}^{n} k \cdot 2^{-k} + \sum_{k=1}^{n} 2^{-k} \right) , \end{aligned}$$

so that by multiplying both sides by 2 we get

$$2S_n + (n+1) \cdot 2^{-n} = 1 + S_n + \sum_{k=1}^{n} 2^{-k} . \tag{3}$$

As the last summand on the right-hand side of (3) is $1 - 2^{-n}$, we get

$$S_n = 2 - (n+2) \cdot 2^{-n} .$$

- Discrete calculus:

  We look at $k \cdot 2^{-k}$ as an object of the form $u \Delta v$, where $u(x) = x$ (so that $\Delta u(x) = 1$) and $\Delta v(x) = 2^{-x}$. Recall that $\Delta c^x = (c-1)c^x$ for $c > 0$: which means that

  $$\Delta 2^{-x} = \Delta \left(\frac{1}{2} \right)^x = \left(\frac{1}{2} - 1 \right) \left(\frac{1}{2} \right)^x = -\frac{1}{2} \cdot 2^{-x} .$$

  To have $\Delta v(x) = 2^{-x}$ we must then set $v(x) = -2 \cdot 2^{-x}$. If we make the additional observation that $\sum_{1 \leqslant k \leqslant n} k \cdot 2^{-k} = \sum_{0 \leqslant k \leqslant n} k \cdot 2^{-k}$, we can compute:

$$\begin{aligned}
\sum_{1\leqslant k\leqslant n} k\cdot 2^{-k} &= \sum_0^{n+1} x\cdot\left(\frac{1}{2}\right)^{x}\,\delta x\\
&= -2x\cdot 2^{-x}\Big|_0^{n+1} - \sum_0^{n+1}(-2)\left(\frac{1}{2}\right)^{x+1}\,\delta x\\
&= -(n+1)\cdot 2^{-n} + \sum_0^{n+1}\left(\frac{1}{2}\right)^{x+1}\,\delta x\\
&= -(n+1)\cdot 2^{-n} + \sum_{k=0}^{n} 2^{-k}\\
&= -(n+1)\cdot 2^{-n} + \left(1+\sum_{k=1}^{n} 2^{-k}\right)\\
&= -(n+1)\cdot 2^{-n} + 1 + 1 - 2^{-n}\\
&= 2-(n+2)\cdot 2^{-n}\,,
\end{aligned}$$

which is the same result we had found by the perturbation method.

Then $\sum_{k\geqslant 1} k\cdot 2^{-k} = \lim_{n\to\infty}\sum_{1\leqslant k\leqslant n} k\cdot 2^{-k} = 2$.

Exercise 4

1. Prove that, for every $n\geqslant 1$,

$$H_n \leqslant 1 + \lfloor \lg n\rfloor\ , \tag{4}$$

   where lg is the base-2 logarithm.

2. Use the inequality (4) to evaluate the infinite sum:

$$\sum_{k\geqslant 1} k^{\underline{-2}}\,H_k\,. \tag{5}$$

Important: Point 2 can be solved without having solved point 1, as it only asks to use the inequality (4), not to have proven it.

Solution. For $n \geqslant 1$ let $m = \lfloor \lg n \rfloor$, so that $2^m \leqslant n \leqslant 2^{m+1} - 1$. Then

$$\begin{aligned} H_n &\leqslant \sum_{k=1}^{2^{m+1}-1} \frac{1}{k} \\ &= \sum_{j=0}^{m} \sum_{k=2^j}^{2^{j+1}-1} \frac{1}{k} \\ &\leqslant \sum_{j=0}^{m} \sum_{k=2^j}^{2^{j+1}-1} \frac{1}{2^j} \\ &= \sum_{j=0}^{m} 1 = m + 1 = 1 + \lfloor \lg n \rfloor \ . \end{aligned}$$

Let now $u(x) = H_x$ and $v(x) = -x^{\underline{-1}} = -1/(x+1)$, so that $\Delta u(x) = \frac{1}{x+1} = x^{\underline{-1}}$ and $\Delta v(x) = x^{\underline{-2}}$. Then for every $n \geqslant 2$:

$$\begin{aligned} \sum_{1 \leqslant k < n} k^{\underline{-2}} H_k &= \sum_{1}^{n} u(x) \, \Delta v(x) \, \delta x \\ &= -x^{\underline{-1}} H_x \Big|_1^n - \sum_1^n E v(x) \, \Delta u(x) \, \delta x \\ &= -\frac{1}{n+1} \cdot H_n + \frac{1}{2} + \sum_1^n (x+1)^{\underline{-1}} x^{\underline{-1}} \, \delta x \\ &= \frac{1}{2} - \frac{H_n}{n+1} + \sum_1^n x^{\underline{-2}} \, \delta x \\ &= \frac{1}{2} - \frac{H_n}{n+1} - x^{\underline{-2}} \Big|_1^n \\ &= \frac{1}{2} - \frac{H_n}{n+1} - \frac{1}{n+1} + \frac{1}{2} \\ &= 1 - \frac{H_n + 1}{n+1} \, . \end{aligned}$$

Because of the inequality (4), the second summand vanishes for $n \to \infty$. We can then conclude that:

$$\sum_{k \geqslant 1} k^{\underline{-2}} H_k = 1 \, .$$

Exercise 5

Prove that $\left\lceil x - \frac{1}{2} \right\rceil \leqslant \left\lfloor x + \frac{1}{2} \right\rfloor$ for every $x \in \mathbb{R}$, and give a closed formula for the difference.

Solution. The closed interval $\left[x - \frac{1}{2}, x + \frac{1}{2}\right]$ contains two integers if $\{x\} = x - \lfloor x \rfloor = \frac{1}{2}$, otherwise it contains a single integer. In this second case, such single integer must be the common value of $\left\lceil x - \frac{1}{2} \right\rceil$ and $\left\lfloor x + \frac{1}{2} \right\rfloor$; otherwise, $x - \frac{1}{2}$ and $x + \frac{1}{2}$ are both integer, so they coincide with both their floors and their ceilings, and the former is smaller than the latter. Then

$$\left\lfloor x + \frac{1}{2} \right\rfloor - \left\lceil x - \frac{1}{2} \right\rceil = \left[x - \lfloor x \rfloor = \frac{1}{2} \right] .$$

Exercise 6

Prove that $n^{13} - n$ is divisible by 105 for every positive integer n.

Solution. As $105 = 3 \cdot 5 \cdot 7$ as a product of (powers of) primes, $n^{13} - n$ is divisible by 105 if and only if it is divisible by 3, 5, and 7. Write $n^{13} - n = n \cdot (n^{12} - 1)$: to apply Fermat's little theorem with prime p, we must collect a factor $n^p - n$ from $n^{13} - n$, or equivalently, a factor $n^{p-1} - 1$ from $n^{12} - 1$. For $p = 3$ we must show that $n^{12} - 1$ is divisible by $n^2 - 1$: but this is true, because

$$n^{12} - 1 = (n^2)^6 - 1 = (n^2 - 1)(n^{10} + n^8 + n^6 + n^4 + n^2 + 1) .$$

Similarly, for $p = 5$ we must show that $n^{12} - 1$ is divisible by $n^4 - 1$: which is the case, because

$$n^{12} - 1 = (n^4)^3 - 1 = (n^4 - 1)(n^8 + n^4 + 1) .$$

Finally, for $p = 7$ we must show that $n^{12} - 1$ is divisible by $n^6 - 1$: which is true, because $n^{12} - 1 = (n^6 - 1)(n^6 + 1)$.

Exercise 7

Prove that $n^{21} - n^{19} - n^3 + n$ is divisible by 114 for every integer $n \geqslant 1$.

Solution. As $114 = 2 \cdot 3 \cdot 19$ as a product of (powers of) primes, we must prove that $n^{21} - n^{19} - n^3 + n$ is divisible by 2, 3, and 19 for every $n \geqslant 1$. Factoring the polynomial, we get:

$$n^{21} - n^{19} - n^3 + n = n \cdot (n^{20} - n^{18} - n^2 + 1) = n \cdot (n^{18} - 1) \cdot (n^2 - 1)\,.$$

This decomposition tells us that $n^{21} - n^{19} - n^3 + n$ is divisible by $n^{19} - n$, which in turn is divisible by 19 because of Fermat's last theorem. Moreover, as $n^2 - 1 = (n-1)(n+1)$, the number $n^{21} - n^{19} - n^3 + n$ always has the three consecutive factors $n - 1$, n, and $n + 1$: of those, exactly one is a multiple of 3, and at least one is even.