ITT9132 Concrete Mathematics
Exercises from Week 10

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Higher order differences

Recall the definition of the forward difference:

$$\Delta f(x) = f(x+1) - f(x)$$

What will be the difference of the difference? Well:

$$\begin{aligned}
\Delta^2 f(x) &= (\Delta f)(x+1) - (\Delta f)(x) \\
&= (f(x+1+1) - f(x+1)) - (f(x+1) - f(x)) \\
&= f(x+2) - 2f(x+1) + f(x)
\end{aligned}$$

And the difference of the above? We now know the trick:

$$\begin{aligned}
\Delta^3 f(x) &= (\Delta^2 f)(x+1) - (\Delta^2 f)(x) \\
&= (f(x+2+1) - 2f(x+1+1) + f(x+1)) \\
&\quad -(f(x+2) - 2f(x+1) + f(x)) \\
&= f(x+3) - 3f(x+2) + 3f(x+1) - f(x)
\end{aligned}$$

However, this is dangerously similar to a class of equations we already know:

$$\begin{aligned}
(x-1)^2 &= x^2 - 2x + 1 \\
(x-1)^3 &= x^3 - 3x^2 + 3x - 1
\end{aligned}$$

and so on. In fact, the following can be proved by induction:

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) \qquad (1)$$

The formula is true for $n = 1, 2, 3$. Suppose it is true for a given $n \geqslant 1$: then,

$$\begin{aligned}
\Delta^{n+1} f(x) &= \Delta\left(\Delta^n f(x)\right) \\
&= \Delta\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k)\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k+1) - \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) \\
&= \binom{n}{n} f(x+n+1) + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} f(x+k+1) \\
&\quad - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} f(x+k) - \binom{n}{0} (-1)^n f(x) \\
&= \binom{n+1}{n+1} f(x+n+1) \\
&\quad + \sum_{k=1}^{n} \left(\binom{n}{k-1} (-1)^{n-(k-1)} - \binom{n}{k} (-1)^{n-k}\right) f(x+k) \\
&\quad + \binom{n+1}{0} (-1)^{n+1} f(x) \\
&= \binom{n+1}{n+1} f(x+n+1) \\
&\quad + \sum_{k=1}^{n} \left(\binom{n}{k-1} (-1)^{n+1-k} + \binom{n}{k} (-1)^{n+1-k}\right) f(x+k) \\
&\quad + \binom{n+1}{0} (-1)^{n+1} f(x) \\
&= \binom{n+1}{n+1} f(x+n+1) \\
&\quad + \sum_{k=1}^{n} \binom{n+1}{k} (-1)^{n+1-k} f(x+k) \\
&\quad + \binom{n+1}{0} (-1)^{n+1} f(x) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} f(x+k)
\end{aligned}$$

Exercise 5.1

Explain why it is easy to evaluate 11^4 for those who know binomial coefficients.

Solution. $11 = 1 + 10$, so by the binomial theorem

$$\begin{aligned} 11^4 &= (1+10)^4 \\ &= \binom{4}{0} \cdot 1^0 \cdot 10^{4-0} + \binom{4}{1} \cdot 1^1 \cdot 10^{4-1} + \binom{4}{2} \cdot 1^2 \cdot 10^{4-2} \\ &\quad + \binom{4}{3} \cdot 1^3 \cdot 10^{4-3} + \binom{4}{4} \cdot 1^4 \cdot 10^{4-4} \\ &= 10000 + 4000 + 600 + 40 + 1 \\ &= 14641 \end{aligned}$$

Exercise 5.2

Find the values of k for which $\binom{n}{k}$ is a maximum. Prove the answer.

Solution. Let $f(k) = \binom{n}{k}$. Then

$$\begin{aligned} \Delta f(k) &= \binom{n}{k+1} - \binom{n}{k} \\ &= \frac{n-k}{k+1}\binom{n}{k} - \binom{n}{k} \\ &= \left(\frac{n-k}{k+1} - 1\right)\binom{n}{k} \end{aligned}$$

The right-hand side has the sign of the term in parentheses, which is the same as that of $n - 1 - 2k$. This means that $\binom{n}{k+1}$ is greater than $\binom{n}{k}$ if $k < (n-1)/2$, and smaller than $\binom{n}{k}$ if $k > (n-1)/2$. Therefore $f(k) = \binom{n}{k}$ is maximum when k is either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$.

The same result can be achieved by considering the ratio $\binom{n}{k+1}/\binom{n}{k}$ instead of the difference $\binom{n}{k+1} - \binom{n}{k}$. In this case:

$$\begin{aligned} \frac{\binom{n}{k+1}}{\binom{n}{k}} &= \frac{\dfrac{n^{\underline{k+1}}}{(k+1)!}}{\dfrac{n^{\underline{k}}}{k!}} \\ &= \frac{n^{\underline{k+1}}}{n^{\underline{k}}} \cdot \frac{k!}{(k+1)!} \\ &= \frac{n^{\underline{k}} \cdot (n-k)}{n^{\underline{k}}} \cdot \frac{k!}{(k+1)k!} \\ &= \frac{n-k}{k+1} \,. \end{aligned}$$

Similarly to the difference, such ratio is larger than 1 if and only if $n - k > k + 1$, that is, $k < (n-1)/2$.

Exercise 5.5

Let p be a prime. Prove that $\binom{p}{k} \equiv 0 \pmod{p}$ for $0 < k < p$. Find a consequence about $\binom{p-1}{k}$.

Solution. Recall that $a \equiv b \pmod{m}$ means that $a - b$ is a multiple of m. By definition:

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!} \,.$$

If p is prime and k is neither 0 nor p, there is no way to make the p at numerator disappear by dividing by $k!$.

Now, $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$: since the left-hand side is 0 modulo p, going from $\binom{p-1}{k-1}$ to $\binom{p-1}{k}$ only involves a change of sign modulo p. Since $\binom{p-1}{0} = 1$, we get:

$$\binom{p-1}{k} \equiv (-1)^k \bmod p$$

A recurrence solved with generating functions

Use generating functions to solve the recurrence:

$$\begin{aligned} U_0 &= 1 \,; \\ U_n &= U_{n-1} + n + 3 \quad \text{for } n > 0 \,. \end{aligned} \tag{2}$$

Solution. For $n > 0$ and $z \neq 0$ the recurrence equation can be rewritten:

$$U_n z^n = U_{n-1} z^n + n z^n + 3 z^n \,.$$

Let $U(z) = \sum_{n \geqslant 0} U_n z^n$ be the generating function of the sequence $\langle U_n \rangle_{n \geqslant 0}$: by summing over n we get

$$\begin{aligned}
U(z) &= 1 + \sum_{n \geqslant 1} U_n z^n \\
&= 1 + \sum_{n \geqslant 1} U_{n-1} z^n + \sum_{n \geqslant 1} n z^n + \sum_{n \geqslant 1} 3 z^n \\
&= 1 + z \sum_{n \geqslant 0} U_n z^n + \sum_{n \geqslant 1} n z^n + 3z \sum_{n \geqslant 0} z^n \\
&= 1 + z U(z) + \frac{z}{(1-z)^2} + \frac{3z}{1-z} \,,
\end{aligned}$$

which can be rewritten

$$(1-z)U(z) = 1 + \frac{z}{(1-z)^2} + 3 \cdot \frac{z}{1-z} \,,$$

which in turn yields

$$U(z) = \frac{1}{1-z} + \frac{z}{(1-z)^3} + 3 \cdot \frac{z}{(1-z)^2} \,.$$

We know that $1/(1-z) = \sum_{n \geqslant 0} z^n$ and $z/(1-z)^2 = \sum_{n \geqslant 0} n z^n$, so we only need to express $z/(1-z)^3$ as a power series. But as $2/(1-z)^3$ is the derivative of $1/(1-3)^2$,

$$\begin{aligned}
\frac{z}{(1-z)^3} &= \frac{z}{2} \frac{d}{dz} \frac{1}{(1-z)^2} \\
&= \frac{z}{2} \frac{d}{dz} \sum_{n \geqslant 1} n z^{n-1} \\
&= \frac{z}{2} \sum_{n \geqslant 2} n(n-1) z^{n-2} \\
&= \frac{1}{2} \sum_{n \geqslant 1} n(n-1) z^{n-1} \\
&= \frac{1}{2} \sum_{n \geqslant 0} (n+1) n z^n \,.
\end{aligned}$$

We can then rewrite our equality as:

$$\sum_{n\geqslant 0} U_n z^n = \sum_{n\geqslant 0} z^n + \frac{1}{2}\sum_{n\geqslant 0} \frac{n(n+1)}{2} z^n + 3\sum_{n\geqslant 0} n z^n \,,$$

and conclude that $U_n = 1 + \dfrac{n(n+1)}{2} + 3n$ for every $n \geqslant 0$.

Exercise 5.7

Prove equality (5.34): for every $r \in \mathbb{C}$ and $k \in \mathbb{N}$,

$$r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}$$

Is the equality true also when $k < 0$?

Solution. We have:

$$\begin{aligned}
r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} &= r(r-1)\cdots(r-k+1)\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\left(r-\frac{2k+1}{2}\right) \\
&= \frac{1}{2^{2r}}2r(2r-2)\cdots(2r-2k-2)(2r-1)(2r-3)\cdots(2r-2k-1) \\
&= \frac{1}{2^{2r}}2r(2r-1)(2r-2)(2r-3)\cdots(2r-2k-2)(2r-2k-1) \\
&= \frac{(2r)^{\underline{2k}}}{2^{2k}}\,.
\end{aligned}$$

Now, for $m > 0$ and $r \in \mathbb{C}$ it is:

$$r^{\underline{-m}} = \frac{1}{(r+1)^{\overline{m}}} = \frac{1}{(-1)^m(-r-1)^{\underline{m}}} = \frac{(-1)^m}{(-r-1)^{\underline{m}}}$$

Then for $k < 0$ and $m = -k = |k|$ equality (5.34) becomes:

$$\frac{(-1)^m}{(-r-1)^{\underline{m}}}\,\frac{(-1)^m}{\left(-r+\frac{1}{2}-1\right)^{\underline{m}}} = 2^{2m}\,\frac{(-1)^{2m}}{(-2r-1)^{\underline{m}}}\,,$$

which is simply (5.34) with (attention!) $-r-1/2$ in place of r, $m = |k|$ in place of k, and the roles of the numerators and denominators swapped!

Exercise 5.17

Find a simple relation between $\binom{2n-1/2}{n}$ and $\binom{2n-1/2}{2n}$.

Solution. Let's follow the suggestion of the book, which tells us that:

$$\binom{2n-1/2}{n} = \frac{1}{2^{2n}}\binom{4n}{2n} \quad \text{and} \quad \binom{2n-1/2}{2n} = \frac{1}{2^{4n}}\binom{4n}{2n}.$$

Together, the two give the formula:

$$\binom{2n-1/2}{n} = 2^{2n}\binom{2n-1/2}{2n}.$$

But how to prove those two? Well, recall equality (5.34):

$$r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}$$

For $r = 2n$ and $k = n$ this becomes:

$$(2n)^{\underline{n}}\left(2n-\frac{1}{2}\right)^{\underline{n}} = \frac{(4n)^{\underline{2n}}}{2^{2n}}$$

that is,

$$n!\binom{2n}{n}\cdot n!\binom{2n-1/2}{n} = \frac{(2n)!}{2^{2n}}\binom{4n}{2n}$$

But $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$, so the above becomes:

$$(2n)!\binom{2n-1/2}{n} = \frac{((2n)!)^2}{2^{2n}}\binom{4n}{2n},$$

which is equivalent to the first equality of the hint. For $r = k = 2n$ we have instead:

$$(2n)^{\underline{2n}}\left(2n-\frac{1}{2}\right)^{\underline{2n}} = \frac{(4n)^{\underline{4n}}}{2^{4n}}$$

Now, for every $m \geqslant 0$ it is $m^{\underline{m}} = m!$, so we can rewrite:

$$(2n)!\left(2n-\frac{1}{2}\right)^{\underline{2n}} = \frac{(4n)!}{2^{4n}},$$

which, by dividing both terms by $((2n)!)^2$, becomes:

$$\binom{2n-1/2}{2n} = \frac{1}{2^{4n}}\binom{4n}{2n},$$

which is the second equality from the hint.