# ITT9132 Concrete Mathematics
# Exercises from Week 11

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## Exercise 6.2

There are $m^n$ functions from a set of $n$ elements to a set of $m$ elements. How many of them range over exactly $k$ different function values?

**Solution.** Suppose $A$ has $n$ elements, $B$ has $m$, and $f : A \to B$ takes exactly $k$ values $b_1, \ldots, b_k$. Then $P = \{f^{-1}(b_i) \mid 1 \leqslant i \leqslant k\}$ is a partition of $A$ in $k$ *nonempty* subsets; moreover, $f$ is completely determined by $P$ and the $b_i$'s.

We have $\left\{ n \atop k \right\}$ ways of choosing $P$. We have $m^{\underline{k}}$ ways of choosing the $b_i$'s. Therefore, we have

$$\left\{ n \atop k \right\} \cdot m^{\underline{k}}$$

ways of constructing $f$.

## Exercise 6.11

Compute $\sum_k (-1)^k \left[ n \atop k \right]$.

**Solution.** We know from the textbook that

$$\sum_k \left[ n \atop k \right] x^k = x^{\overline{n}}$$

For $x = -1$ we get

$$\sum_k (-1)^k \left[ n \atop k \right] = (-1)^{\overline{n}}$$

This is 1 for $n = 0$, $-1$ for $n = 1$, and 0 otherwise. A one-liner is:

$$\sum_k (-1)^k \left[ n \atop k \right] = [n = 0] - [n = 1] \ .$$

## Exercise 6.16

What is the general solution of the double recurrence

$$\begin{array}{llr} A_{n,0} = a_n[n \geqslant 0]\,; & A_{0,k} = 0\,, \quad \text{if } k > 0\,; \\ A_{n,k} = kA_{n-1,k} + A_{n-1,k-1}\,, & k, n \in \mathbb{Z}\,, \end{array} \tag{1}$$

when $k$ and $n$ range over the set of *all* integers?

**Solution.** The double recurrence (1) is linear in the following sense: if $A_{n,k} = U_{n,k}$ is the solution for $a_n = u_n$ and $A_{n,k} = W_{n,k}$ is the solution for $a_n = w_n$, then $A_{n,k} = \lambda U_{n,k} + \mu W_{n,k}$ is the solution for $a_n = \lambda u_n + \mu w_n$. We also know that $A_{n,k} = \left\{ {n \atop k} \right\}$ is the solution for $a_n = [n = 0]$: by linearity, $A_{n,k} = a \cdot \left\{ {n \atop k} \right\}$ is the solution for $a_n = a\,[n = 0]$.

Let us search for the solution of (1) in the more general case $a_n = [n = j]$, where $j$ is an arbitrary integer. This seems difficult, but we observe that (1) displays the following property: if $U_{n,k}$ is the solution for $a_n = u_n$, and $j > 0$, then $A_{n,k} = U_{n-j,k}$ is the solution for $a_n = u_{n-j}\,[n \geqslant j]$: hence, $A_{n,k} = \left\{ {n-j \atop k} \right\}$ is the solution for $a_n = [n = j]$. By linearity, we can conclude that, if $a_n \neq 0$ only for finitely many values of $n$, then

$$A_{n,k} = \sum_{j \geqslant 0} a_j \left\{ {n-j \atop k} \right\} \tag{2}$$

is the solution to (1).

Can we conclude that (2) is the solution to (1) also when infinitely many of the values $a_n$ are nonzero? Yes, because $\left\{ {m \atop k} \right\}$ is zero if $k > m$, thus only the values $a_j$ with $0 \leqslant j \leqslant n - k$ contribute to the sum.

## Exercise 6.28

For $n$ integer, define the $n$th *Lucas number* as $L_n = f_{n+1} + f_{n-1}$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_n$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | ... |

1. Use the repertoire method to find the general solution to the recurrence:

$$\begin{aligned} Q_0 &= \alpha \\ Q_1 &= \beta \\ Q_2 &= Q_{n-1} + Q_{n-2}\ , \ n > 1 \end{aligned}$$

2. Find a closed form for $L_n$ in terms of $\phi$ and $\hat{\phi}$.

**Solution. Point 1.** The Fibonacci numbers satisfy the recurrence with $\alpha = 0$, $\beta = 1$. The Lucas numbers also satisfy the recurrence:

$$L_{n-1} + L_{n-2} = f_n + f_{n-2} + f_{n-1} + f_{n-3} = f_{n+1} + f_{n-1} = L_n$$

We thus only need to reconstruct the initial condition through the system

$$\begin{aligned} xf_0 + yL_0 &= \alpha \\ xf_1 + yL_1 &= \beta \end{aligned}$$

that is,

$$\begin{aligned} 2y &= \alpha \\ x + y &= \beta \end{aligned}$$

which yields $y = \alpha/2, x = \beta - \alpha/2$. Therefore,

$$Q_n = xf_n + yL_n = \frac{\alpha}{2}(L_n - f_n) + \beta f_n$$

**Point 2.** We know that $f_n = (\phi^n - \hat{\phi}^n)/\sqrt{5}$. Then,

$$\begin{aligned} L_n &= \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} \\ &= \frac{\phi^{n-1}(\phi^2 + 1) - \hat{\phi}^{n-1}(\hat{\phi}^2 + 1)}{\sqrt{5}} \end{aligned}$$

But

$$\phi^2 + 1 = \frac{6 + 2\sqrt{5}}{4} + 1 = \frac{5 + \sqrt{5}}{2} = \phi\sqrt{5}$$

and similarly,

$$\hat{\phi}^2 + 1 = \frac{6 - 2\sqrt{5}}{4} + 1 = \frac{5 - \sqrt{5}}{2} = -\hat{\phi}\sqrt{5}$$

Consequently,

$$L_n = \frac{\phi^{n-1}(\phi\sqrt{5}) - \hat{\phi}^{n-1}(-\hat{\phi}\sqrt{5})}{\sqrt{5}} = \phi^n + \hat{\phi}^n$$

## Fibonacci number system

Prove *Zeckendorf's theorem*: every positive integer $n$ has a unique writing

$$n = f_{k_1} + f_{k_2} + \ldots + f_{k_r}$$

as a sum of Fibonacci numbers such that:

1. $k_1 > k_2 > \ldots > k_r \geqslant 2$, and
2. no two $k_i$s are consecutive, that is, for no $i$ it is $k_i = k_{i+1} + 1$.

**Solution.** The thesis is true for $n = 1 = f_2$, $n = 2 = f_3$, $n = 3 = f_4$, and $n = 4 = 3 + 1 = f_4 + f_2$, so we have a good base to proceed with a proof by strong induction.

Suppose the thesis is true for every positive integer $m < n$. Let $k_1$ be the largest integer such that $f_{k_1} \leqslant n$: as $n > 0$, $k_1 \geqslant 2$. If $n = f_{k_1}$ we are done, while if $n = f_{k_1} + 1$ we put $k_2 = 2$ and we are done. Otherwise, $m = n - f_{k_1} < n$ is positive, so by inductive hypothesis it has a unique writing:

$$m = f_{k_2} + f_{k_3} + \ldots + f_{k_r}$$

satisfying our two conditions. Then $k_1$ and $k_2$ are not consecutive: if they were, then $k_2 = k_1 - 1$, and

$$\begin{aligned} n &= f_{k_1} + f_{k_1-1} + f_{k_3} + \ldots + f_{k_r} \\ &= f_{k_1+1} + f_{k_3} + \ldots + f_{k_r} \, , \end{aligned}$$

against our hypothesis that $k_1$ is maximum such that $f_{k_1} \leqslant n$.