Concrete Mathematics
Exercises from Week 12

Silvio Capobianco

Exercise 6.4

Express $1 + 1/3 + \ldots + 1/(2n+1)$ in terms of harmonic numbers.

Solution. If the summands $1/2, 1/4, \ldots, 1/2n$ were present, that would be H_{2n+1}. But they are not there, thus their sum, which is $H_n/2$, has been subtracted from the total. Then $1 + 1/3 + \ldots + 1/(2n+1) = H_{2n+1} - \frac{1}{2}H_n$.

Exercise 6.20

Find a closed form for $\sum_{k=1}^{n} H_k^{(2)}$.

Solution. We rewrite:

$$\begin{aligned}
\sum_{k=1}^{n} H_k^{(2)} &= \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{j^2} \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} \frac{1}{j^2} \\
&= \sum_{j=1}^{n} \frac{n+1-j}{j^2} \\
&= (n+1) \sum_{j=1}^{n} \frac{1}{j^2} - \sum_{j=1}^{n} \frac{1}{j} \\
&= (n+1) H_n^{(2)} - H_n \, .
\end{aligned}$$

Observe the similarity with $\sum_{k=1}^{n} H_k = (n+1)H_n - n$.

Exercise 6.21

Show that if $H_n = a_n/b_n$ where a_n and b_n are integers, the denominator b_n is a multiple of $2^{\lfloor \lg n \rfloor}$. *Hint:* consider the number $2^{\lfloor \lg n \rfloor - 1} H_n - \frac{1}{2}$.

Solution. Call $m = \lfloor \lg n \rfloor$ for brevity. Split H_n as follows:

$$H_n = \sum_{k=1}^{2^m - 1} \frac{1}{k} + \frac{1}{2^m} + \sum_{k=2^m+1}^{n} \frac{1}{k} \,.$$

In both summations, the maximum power of 2 which divides k is 2^{m-1}: hence, $2^{m-1} H_n$ is the sum of $n - 1$ rational numbers with odd divisors, plus $1/2$. The writing of $2^{m-1} H_n$ as an irreducible fraction a/b must then have $t = 2d$ for some d odd: consequently, the writing of H_n as an irreducible fraction must have denominator $2^m d$. The denominator in any writing of H_n as a fraction (not necessarily an irreducible one) must then be a multiple of 2^m.

Exercise 6.22

Let z be a complex number. Consider the sum

$$\sum_{k \geqslant 1} \left(\frac{1}{k} - \frac{1}{k+z} \right) \qquad (1)$$

1. Prove that (1) converges for every complex number z except the negative integers.

2. Observe that (1) equals H_n when $z = n$ is a positive integer.

Solution. If z is a negative integer, then some of the summands are undefined. Otherwise, the general term is

$$a_k = \frac{1}{k} - \frac{1}{k+z} = \frac{k+z-k}{k(k+z)} = \frac{z}{k^2 + kz}$$

By the second triangle inequality, $|a - b| \geqslant ||a| - |b||$: for $a = k$, $b = -z$, and $k > |z|$ we have $|k^2 + kz| \geqslant k \cdot |k - |z|| > (k - |z|)^2$. Then $\sum_{k \geqslant 1} |a_k|$ converges by comparison, and (1) converges.

If $z = n$ is a positive integer, then the m-th partial sum is

$$\begin{aligned}\sum_{k=1}^{m}\left(\frac{1}{k}-\frac{1}{k+n}\right) &= \sum_{k=1}^{m}\frac{1}{k}-\sum_{j=n+1}^{n+m}\frac{1}{j} \\ &= H_m-(H_{m+n}-H_n) \\ &= H_n-(H_{m+n}-H_m) \\ &= H_n-\frac{1}{m+1}-\frac{1}{m+2}-\ldots-\frac{1}{m+n}\,,\end{aligned}$$

that is, H_n minus n summands that vanish for $m \to \infty$.

Exercise 6.23

Equation (6.81) gives the coefficients of $z/(e^z-1)$, when expanded in powers of z. What are the coefficients of $z/(e^z+1)$? *Hint:* Consider the identity $(e^z+1)(e^z-1) = e^{2z}-1$.

Solution. We use the hint to find coefficients A and B such that:

$$\frac{1}{e^{2z}-1}=\frac{A}{e^z+1}+\frac{B}{e^z-1}=\frac{(A+B)e^z+(B-A)}{e^{2z}-1}\,.$$

This yields $A+B=0$ and $B-A=1$, so $A=-1/2$ and $B=1/2$, and:

$$\frac{1}{e^{2z}-1}=-\frac{1}{2}\cdot\frac{1}{e^z+1}+\frac{1}{2}\cdot\frac{1}{e^z-1}\,,$$

which in turn yields:

$$\frac{z}{e^z+1}=\frac{z}{e^z-1}-\frac{2z}{e^{2z}-1}\,.$$

This is especially convenient, because the second summand on the right-hand side is simply the first one computed in $2z$ instead of z. Then:

$$\begin{aligned}\frac{z}{e^z+1} &= \sum_{n\geqslant 0}\frac{B_n}{n!}z^n-\sum_{n\geqslant 0}\frac{B_n}{n!}(2z)^n \\ &= \sum_{n\geqslant 0}\frac{(1-2^n)B_n}{n!}z^n\,.\end{aligned}$$

Exercise 6.27

Prove the gcd law (6.111) for the Fibonacci numbers.

Solution. We are required to prove that, for positive m and n,

$$\gcd(f_m, f_n) = f_{\gcd(m,n)} \,. \tag{2}$$

We first prove (2) for $n = m + 1$. As $f_{m+1} = f_m + f_{m-1}$, a common divisor d of f_{m+1} and f_m should also divide f_{m-1}: then it would also divide $f_{m-2} = f_m - f_{m-1}$, and f_{m-3} as well, and so on up to $f_1 = 1$. Thus, $\gcd(f_m, f_{m+1}) = 1 = f_1 = f_{\gcd(m,m+1)}$.

Let us now prove the general case. Suppose for convenience $n > m$. By the generalized Cassini identity, $f_n = f_m f_{n-m+1} + f_{m-1} f_{n-m}$: then,

$$\gcd(f_m, f_n) = \gcd(f_n \bmod f_m, f_m) = \gcd(f_{m-1} f_{n-m}, f_m) = \gcd(f_{n-m}, f_m) \,,$$

because consecutive Fibonacci numbers are relatively prime. But we can continue subtracting m until $n - km$ becomes smaller than m, that is, until $k = \lfloor n/m \rfloor$ and $n - km = n \bmod m$: then

$$\gcd(f_m, f_n) = \gcd(f_{n \bmod m}, f_m) \,.$$

But the equality above means precisely that we can run the Euclidean algorithm on the indices of the Fibonacci numbers, instead of the Fibonacci numbers themselves! The thesis clearly follows.