Concrete Mathematics
Exercises from Week 13

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Exercise 7.1

Consider a strip of $n \times 2$ (n in horizontal, 2 in vertical) square units. To tile such a strip we have a certain number of domino (1×2) tiles, red and blue. Red tiles can only be laid horizontally, while blue tiles can only be laid vertically.

1. Let t_n be the number of ways in which an $n \times 2$ strip can be tiled. Prove that $t_n = f_{n+1}$ for every $n \geqslant 0$, where f_n is the nth Fibonacci number.

2. An eccentric collector of $n \times 2$ domino tilings pays \$4 for each blue domino and \$1 for each red domino. How many tilings are worth exactly \$$m$ by this criterion? For example, when $m = 6$ there are three solutions: one blue and two red, two red and a blue, and six red.

Solution. First, let's compute the first values of t_n:

- $t_0 = 1$, because the only way to tile an empty 0×2 strip is by doing nothing.

- $t_1 = 1$, because the only way to tile a 1×2 strip is to put a blue tile.

- $t_2 = 2$, because the only ways to tile a 1×2 strip are with either two blue tiles, or two red.

- $t_2 = 3$, because the only ways to tile a 1×2 strip are with either three blue tiles, or one blue and two red, or two red and one blue.

This finite sequence $\langle 1, 1, 2, 3\rangle$ is the same as that of the first four Fibonacci numbers with positive integers, so the thesis of point 1 doesn't peregrine. If we manage to prove that $t_n = t_{n-1} + t_{n-2}$ for every $n \geqslant 2$, then the thesis will be proved because of the uniqueness of the solution of a linear recurrence with given initial condition. But for $n \geqslant 2$ we have two mutually exclusive ways of constructing an $n \times 2$ tiling:

1. either by taking an $(n-1) \times 2$ tiling and putting a blue tile at the end,

2. or by taking an $(n-2) \times 2$ tiling and putting two red tiles at the end.

We have thus shown that $t_n = t_{n-1} + t_{n-2}$ for every $n \geqslant 2$, which together with $t_0 = 1 = f_1$ and $t_1 = 1 = f_2$ proves $t_n = f_{n+1}$ for every $n \geqslant 0$.

Now, let u_n be the number of tilings which are paid n dollars and let $U(z)$ be the generating function of the sequence $\langle u_n\rangle$. As any tiling must have evenly many red tiles, and the only tiling which is not paid is the empty one,it is $u_m = 0$ for every m odd, and we expect $U(z) = T(z^2)$ for some function $T(z)$, and $U(0) = u_m = 0$.

So let's simplify the counting and put $v_n = u_{2n}$. The first values are $v_0 = 1$, because we can only make 0 dollars by not doing anything, and $v_1 = 1$, because we can only make 2 dollars with a 2×2 tiling made of two horizontal tiles. For $n \geqslant 2$ the only ways to make $2n$ dollars is to either take a tiling worth $2n-2$ dollars and add two red tiles at the end, or take a tiling worth $2n-4$ dollars and add one red tile at the end: that is,

$$v_n = v_{n-1} + v_{n-2} \quad \forall n \geqslant 2$$

But this is again the Fibonacci recurrence with initial condition $t_0 = 1 = f_1$ and $t_1 = 1 = f_2$: which has the solution $v_n = f_{n+1}$. Then:

$$\begin{aligned} u_n &= \begin{cases} f_{m+1} & \text{if } n = 2m\,, \\ 0 & \text{if } n = 2m+1 \end{cases} \\ &= f_{\lfloor n/2 \rfloor + 1} \cdot [n \text{ is even}]\,, \end{aligned}$$

because if b is a bit and $n = 2m + b$, then $m = \lfloor m/2 \rfloor$.

Exercise 7.6

Solve the recurrence (7.32):

$$\begin{aligned} g_0 &= 1\,, \\ g_1 &= 1\,, \\ g_n &= g_{n-1} + 2g_{n-2} + (-1)^n \quad \forall n \geqslant 2\,. \end{aligned}$$

in two different ways:

1. using generating functions and the Rational Expansion Theorem;
2. by the repertoire method.

Solution. Let us start with point 1. If we use the convention that $g_n = 0$ for $n < 0$, the recurrence will have the form:

$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n \cdot [n \geqslant 0] + \text{correction terms} \quad \forall n \in \mathbb{Z}\,.$$

For $n = 0$ we have $1 = (-1)^0$, so we need no correction term. For $n = 1$ we have $g_1 = 1$ but $g_0 = 1$ and $(-1)^1 = -1$, so we need a correction term $+1$. Then:

$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n \cdot [n \geqslant 0] + [n = 1] \quad \forall n \in \mathbb{Z}\,,$$

and the generating function $G(z)$ satisfies the equation:

$$G(z) = zG(z) + z^2 G(z) + \frac{1}{1+z} + z\,,$$

which yields:

$$\begin{aligned} G(z) &= \frac{1}{1 - z - 2z^2} \cdot \left(\frac{1}{1+z} + z \right) \\ &= \frac{1}{(1+z)(1-2z)} \cdot \left(\frac{1 + z + z^2}{1+z} \right) \\ &= \frac{1 + z + z^2}{(1-2z)(1+z)^2} \end{aligned}$$

The Rational Expansion Theorem then tells that it must be:

$$g_n = a \cdot 2^n + (b + cn) \cdot (-1)^n$$

and that, as the denominator $Q(z) = 1 - 3z^2 - 2z^3$ has $Q'(z) = -6z - 6z^2 = -6z(1+z)$ and $Q''(z) = -6 + 12z$, we can immediately compute:

$$\begin{aligned} a &= \frac{1 + 1/2 + 1/4}{0!(1 + 1/2)^2} \\ &= \frac{7/4}{9/4} = \frac{7}{9}, \\ c &= \frac{1 - 1 + (-1)^2}{1!(1 - 2/(-1))} = \frac{1}{3}. \end{aligned}$$

To find b, we put $n = 0$ and obtain:

$$\frac{7}{9} \cdot 2^0 + \left(\frac{1}{3} \cdot 0 + b \right) = g_0 = 1,$$

whence $b = \frac{2}{9}$. In conclusion:

$$g_n = \frac{7}{9} \cdot 2^n + \left(\frac{n}{3} + \frac{2}{9} \right) \cdot (-1)^n \text{ for every } n \geqslant 0.$$

Now, to the repertoire method. The recurrence (7.32) is a special case of the general recurrence:

$$\begin{aligned} g_0 &= \alpha, \\ g_1 &= \beta, \\ g_n &= g_{n-1} + 2g_{n-2} + (-1)^n \cdot \gamma \ \ \forall n \geqslant 2 \end{aligned}$$

when $\alpha = \beta = \gamma = 1$. Let us look for a solution of the form

$$g_n = \alpha \cdot A(n) + \beta \cdot B(n) + \gamma \cdot C(n)$$

for suitable functions $A(n)$, $B(n)$, and $C(n)$.

First, we try to exploit the dependence on the *double* of the second previous step and set $g_n = 2^n$: then the recurrence becomes

$$\begin{aligned} 2^0 &= \alpha, \\ 2^1 &= \beta, \\ 2^n &= 2^{n-1} + 2 \cdot 2^{n-2} + (-1)^n \cdot \gamma \ \ \forall n \geqslant 2, \end{aligned}$$

which is satisfied for every $n \geqslant 0$ by choosing $\alpha = 1$, $\beta = 2$, $\gamma = 0$.

Next, we try to exploit dependence on the sign of n and set $g_n = (-1)^n$: then the recurrence becomes

$$\begin{aligned} 1 &= \alpha\,, \\ -1 &= \beta\,, \\ (-1)^n &= (-1)^{n-1} + 2\cdot(-1)^{n-2} + (-1)^n \cdot \gamma \ \ \forall n \geqslant 2\,, \end{aligned}$$

which is satisfied for every $n \geqslant 0$ by choosing $\alpha = 1$, $\beta = -1$, $\gamma = 0$, as $(-1)^{n-1} + 2\cdot(-1)^{n-2}$ can be rewritten as $(-1)^{n-1}\cdot(1-2)$, which clearly equals $(-1)^n$.

Finally, we try to exploit the presence of the summand $(-1)^n$, and set $g_n = (-1)^n \cdot n$, which is the second simplest function to depend on $(-1)^n$; then the recurrence becomes

$$\begin{aligned} 0 &= \alpha\,, \\ -1 &= \beta\,, \\ (-1)^n n &= (-1)^{n-1}(n-1) + 2\cdot(-1)^{n-2}(n-2) + (-1)^n \cdot \gamma \ \ \forall n \geqslant 2\,, \end{aligned}$$

which is satisfied for every $n \geqslant 0$ by putting $\alpha = 0$, $\beta = -1$, and $\gamma = 3$ as the recurrence equation can be rewritten as $n = (1-n) + 2(n-2) + \gamma$ by dividing it by $(-1)^n$.

The three functions $A(n)$, $B(n)$, $C(n)$ are thus the solutions of the system:

$$\begin{array}{llll} A(n) & +2B(n) & & = \quad 2^n \\ A(n) & -B(n) & & = \quad (-1)^n \\ & -B(n) & +3C(n) & = \quad (-1)^n \cdot n \end{array}$$

Adding twice the second equation to the first one yields

$$A(n) = \frac{2^n + 2\cdot(-1)^n}{3}\,;$$

subtracting the second equation from the first one yields

$$B(n) = \frac{2^n - (-1)^n}{3}\,;$$

replacing $B(n)$ in the third equation yields

$$C(n) = \frac{2^n + (-1)^n \cdot (3n-1)}{9}\,.$$

By setting $\alpha = \beta = \gamma = 1$ we finally get the solution to the original recurrence:

$$\begin{aligned} g_n &= \frac{2^n + 2 \cdot (-1)^n}{3} \\ &\quad + \frac{2^n - (-1)^n}{3} \\ &\quad + \frac{2^n + (-1)^n \cdot (3n - 1)}{9} \\ &= \frac{(3 + 3 + 1) \cdot 2^n + (6 - 3 - 1 + 3n) \cdot (-1)^n}{9} \\ &= \frac{7}{9} \cdot 2^n + \left(\frac{n}{3} + \frac{2}{9} \right) (-1)^n . \end{aligned}$$

As a final consideration, the repertoire method is great for finding solutions of *families* of linear recurrences, but for *single* linear recurrences the Rational Expansion Theorem applied to generating functions is much more straightforward.