ITT9132 Concrete Mathematics
Exercises from Week 14

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Exercise RET2

Solve the recurrence

$$g_n = 6g_{n-1} - 9g_{n-2} \quad \forall n \geqslant 2 \tag{1}$$

with the initial conditions $g_0 = 1$, $g_1 = 9$.

Solution. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$, with the convention that $g_n = 0$ if $n < 0$. The recurrence $g_n z^n = 6g_{n-1}z^n - 9g_{n-2}z^n$ holds for every $n < 0$ and $n \geqslant 2$; for $n = 0$ we must have $1 = g_0 = 6g_{-1} - 9g_{n-2} + 1$; for $n = 1$ we must have $9 = g_1 = 6g_0 - 9g_{-1} + 3$. Then,

$$\sum_n g_n z^n = 6 \sum_n g_{n-1} z^n - 9 \sum_n g_{n-2} z^n + \sum_n [n = 0]\, z^n + 3 \sum_n [n = 1]\, z^n \,,$$

that is,

$$G(z) = 6zG(z) - 9z^2 G(z) + 1 + 3z \; :$$

which yields

$$G(z) = \frac{1 + 3z}{1 - 6z + 9z^2} = \frac{1 + 3z}{(1 - 3z)^2} \,.$$

In the notation of the Rational Expansion Theorem, we have $P(z) = 1 + 3z$, $Q(z) = (1 - 3z)^2$, $\rho_1 = 3$, $d_1 = 2$. Therefore, $Q'(z) = -6 + 18z$, $Q''(z) = 18$, and $g_n = (a_1 n + c_1) \cdot 3^n$, where

$$a_1 = \frac{(-3)^2 \cdot (1 + 3/3) \cdot 2}{18} = 2 \,.$$

For $n = 0$ we find $1 = g_0 = (0 + c_1) \cdot 1$, yielding $c_1 = 1$. Therefore,

$$g_n = (2n + 1) \cdot 3^n \,.$$

Exercise RET3

Solve the recurrence

$$g_n = 3g_{n-1} - 4g_{n-3} \quad \forall n \geqslant 3 \tag{2}$$

with the initial conditions $g_0 = 0$, $g_1 = 1$, $g_2 = 3$.

Solution. Observe that (2) is a recurrence of the *third* order, since g_n depends on g_{n-1} and g_{n-3}: therefore, we need *three* initial conditions.

Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$, with the convention that $g_n = 0$ if $n < 0$. The recurrence $g_n = 3g_{n-1} - 4g_{n-3}$ holds for every $n < 0$ and $n \geqslant 3$; for $n = 0$ we have $0 = g_0 = 3g_{-1} - 4g_{-3}$; for $n = 1$ we have $1 = g_1 = 3g_0 - 4g_{-2} + 1$; for $n = 2$ we have $3 = g_2 = 3g_1 - 4g_{-1}$. Then,

$$\sum_n g_n z^n = 3\sum_n g_{n-1} z^n - 4\sum_n g_{n-3} z^n + \sum_n [n = 1]\, z^n\,,$$

that is,

$$G(z) = 3zG(z) - 4z^3 G(z) + z\ :$$

which yields

$$G(z) = \frac{z}{1 - 3z + 4z^3}\,.$$

We observe that $Q(1/2) = Q(-1) = 0$: and in fact, if we divide $Q(z)$ by $1 + z$, we get $1 - 4z + 4z^2 = (1 - 2z)^2$. Therefore,

$$G(z) = \frac{z}{(1 + z)(1 - 2z)^2}\,.$$

In the notation of the Rational Expansion Theorem, we have $\rho_1 = -1$, $d_1 = 1$, $\rho_2 = 2$, $d_2 = 2$; also, $P(z) = z$ and

$$Q(z) = 1 - 3z + 4z^3\,,$$

from which $Q'(z) = -3 + 12z^2$ and $Q''(z) = 24z$. Then $g_n = a_1 \cdot (-1)^n + (a_2 n + c_2) \cdot 2^n$ for suitable a_1, a_2, c_2, where

$$a_1 = \frac{1^1 \cdot (-1)}{-3 + 12} = -\frac{1}{9}$$

and

$$a_2 = \frac{(-2)^2 \cdot (1/2) \cdot 2}{24 \cdot 1/2} = \frac{1}{3}\,.$$

For $n = 0$ we find $0 = -\frac{(-1)^0}{9} + (0 + c_2) \cdot 1$, yielding $c_2 = 1/9$. Therefore,

$$g_n = \frac{(-1)^{n+1}}{9} + \left(\frac{n}{3} + \frac{1}{9}\right) 2^n = \frac{(-1)^{n+1}}{9} + \frac{3n + 1}{9} \cdot 2^n\,.$$

Exercise 7.7

Solve the recurrence:

$$\begin{aligned} g_0 &= 1 \\ g_n &= g_{n-1} + 2g_{n-2} + \ldots + ng_0 \end{aligned}$$

Solution. Let $G(z)$ be the generating function of the sequence $\langle g_0, g_1, g_2, \ldots \rangle$. The recurrence above tells us that $G(z)$ is the convolution of itself with the generating function of the sequence $\langle 0, 1, 2, \ldots \rangle$, which is $z/(1-z)^2$: except for the first term, which is 1 instead of $0 = 0 \cdot g_0$. Hence,

$$G(z) = 1 + \frac{zG(z)}{(1-z)^2}$$

which rewrites as

$$(1-z)^2 G(z) = (1-z)^2 + zG(z)$$

which yields

$$(1 - 3z + z^2)G(z) = (1-z)^2$$

that is,

$$G(z) = \frac{1 - 2z + z^2}{1 - 3z + z^2} = 1 + \frac{z}{1 - 3z + z^2}$$

The first summand on the right-hand side is clearly the generating function of $\langle [n = 0] \rangle$; the second one is the generating function of $\langle f_{2n} \rangle$, where f_n is the nth Fibonacci number. Therefore, $g_n = f_{2n} + [n = 0]$.

Bonus

Prove that

$$\sum_{k \geqslant 0} f_{2k} z^k = \frac{z}{1 - 3z + z^2} \qquad (3)$$

Solution. We know that

$$\sum_{k \geqslant 0} f_k z^k = \frac{z}{1 - z - z^2}$$

Now, if $G(z) = \sum_{n\geqslant 0} a_n z^n$, then $G(-z) = \sum_{n\geqslant 0} a_n(-z)^n = \sum_{n\geqslant 0}((-1)^n a_n)z^n$. Consequently,

$$\begin{aligned}\frac{G(z)+G(-z)}{2} &= \sum_{n\geqslant 0} \frac{1+(-1)^n}{2} a_n z^n \\ &= \sum_{n\geqslant 0} [n \text{ is even}]\ a_n z^n \\ &= \sum_{n\geqslant 0} a_{2n} z^{2n}\end{aligned}$$

Thus,

$$\sum_{k\geqslant 0} f_{2k} z^{2k} = \frac{1}{2}\left(\frac{z}{1-z-z^2} + \frac{-z}{1+z-z^2}\right) = \frac{z^2}{1-3z^2+z^4}$$

By replacing z^2 with z we retrieve (3).

Exercise 7.11

Let $a_n = b_n = c_n = 0$ for $n < 0$, and

$$A(z) = \sum_n a_n z^n \ \ ; \ \ B(z) = \sum_n b_n z^n \ \ ; \ \ C(z) = \sum_n c_n z^n$$

1. Express $C(z)$ in terms of $A(z)$ and $B(z)$ when $c_n = \sum_{j+2k\leqslant n} a_j b_k$.

2. Express $A(z)$ in terms of $B(z)$ when $nb_n = \sum_{k=0}^{n} 2^k a_k/(n-k)!$

Solution. Point 1. We know that, if $a_n = [z^n]G(z)$, then $\sum_{k\leqslant n} a_k = [z^n]\dfrac{G(z)}{1-z}$. Then we can solve point 1 as soon as we find $G(z)$ such that $[z^n]G(z) = \sum_{j+2k=n} a_j b_k$. But the latter is the coefficient of index n of the convolution of A with a power series whose odd-indexed coefficients are 0, and whose coefficient of index $2k$ is b_k: such function is precisely $B(z^2)$. Therefore,

$$C(z) = \frac{A(z)B(z^2)}{1-z}$$

Point 2. We know that $nb_n = [z^{n-1}]B'(z) = [z^n]zB'(z)$. Moreover, nb_n must be the coefficient of index n of the convolution of $A(2z)$ (because of the 2^k factor) with a power series whose coefficient of index n is $1/n!$: such function is e^z. This means

$$zB'(z) = e^z A(2z)$$

and consequently

$$A(z) = \frac{z}{2} e^{-z/2} B'\left(\frac{z}{2}\right) .$$

Exercise 7.12

How many ways are there to put the numbers $\{1, 2, \ldots, 2n\}$ into a $2 \times n$ array so that rows and columns are in increasing order from left to right and from top to bottom? For example, one solution when $n = 5$ is

$$\begin{pmatrix} 1 & 2 & 4 & 5 & 8 \\ 3 & 6 & 7 & 9 & 10 \end{pmatrix} .$$

Solution. We construct a bijection from the set of $2 \times n$ arrays satisfying our constraints to the set of mountain chains of length $2n$ in the following way: for each i from 1 to $2n$, the segment i goes up if i is on the first row, and down if i is on the second row. (This satisfies the basic sanity check that the upper left corner of the array can only be 1, and the lower right one can only be $2n$.)

That the function is well defined, follows precisely from the fact that the two rows and each column are in increasing order: indeed, this translates into the jth downslope coming no sooner than j upslopes, and there being as many upslopes as downslopes. That the function is bijective, follows from it having an immediate inverse, constructed by writing in sequence each $i \in \{1, \ldots, 2n\}$ in the upper row if the ith slope goes up, and in the lower row if it goes down: this immediately ensures the rows to be sorted low to high, while sorting the columns corresponds to the fact that in a mountain chain the difference between upslopes and downslopes until each point is nonnegative.

The number of ways to put the numbers $\{1, 2, \ldots, 2n\}$ into a $2 \times n$ array so that rows and columns are in increasing order from left to right and from top to bottom, is thus the same as the number of mountain chains of length $2n$: which is notoriously the nth *Catalan number* $C_n = \binom{2n}{n}\frac{1}{n+1}$.

Exercise 7.35

Evaluate the sum $\sum_{0<k<n} 1/k(n-k)$ in two ways:

1. Expand the summand in partial fractions.
2. Treat the sum as a convolution and use generating functions.

Solution. Expanding $1/k(n-k)$ in partial fractions means finding constants A and B such that
$$\frac{1}{k(n-k)} = \frac{A}{k} + \frac{B}{n-k} \ :$$
from $\frac{1}{k} + \frac{1}{n-k} = \frac{n}{k(n-k)}$ we easily get $A = B = \frac{1}{n}$. Then:
$$\sum_{0<k<n} \frac{1}{k(n-k)} = \frac{1}{n} \sum_{0<k<n} \left(\frac{1}{k} + \frac{1}{n-k} \right) = \frac{2}{n} H_{n-1} \, .$$

We can also observe that $g_n = \sum_{0<k<n} \frac{1}{k(n-k)}$ is the term of index n of the convolution of the sequence of generic term $h_n = \frac{1}{n}[n>0]$ with itself. Let $G(z)$ and $H(z)$ be the generating functions of the sequences $\langle g_n \rangle$ and $\langle h_n \rangle$, respectively: we know that $H(z) = \log \frac{1}{1-z}$, so
$$G(z) = H(z)^2 = \left(\log \frac{1}{1-z} \right)^2 . \tag{4}$$

This looks hard to manage until we remember that, if $G(z) = \sum_n g_n z^n$, then $zG'(z) = \sum_n n g_n z^n$. Said, done:
$$\begin{aligned} zG'(z) &= z \frac{d}{dz} \left(\log \frac{1}{1-z} \right)^2 \\ &= z \cdot \left(2 \log \frac{1}{1-z} \right) \cdot \frac{1}{\frac{1}{1-z}} \cdot \frac{1}{(1-z)^2} \\ &= 2z \cdot \left(\frac{1}{1-z} \log \frac{1}{1-z} \right) . \end{aligned}$$

The function in parentheses on the last line is the generating function of the harmonic numbers. (More in general, if $G(z)$ is the generating function of

$\langle g_n \rangle$, then $G(z)/(1-z)$ is the generating function of $\left\langle \sum_{0 \leqslant k \leqslant n} g_k \right\rangle$. Recall our convention that undefined $\cdot$ [False] $= 0$.) By pre-multiplying by z, H_n becomes the coefficient of z^{n+1} instead of z^n. Equating the power series,

$$\sum_n n g_n z^n = 2 \sum_n H_n z^{n+1} = 2 \sum_n H_{n-1} z^n \; :$$

then $n g_n = 2 H_{n-1}$ for every n, which is equivalent to what we had found before.

Lesson learned: if you need to kill a fly, don't use a cannon!