Concrete Mathematics
Exercises from Week 15

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Exercise 9.1

Prove or disprove: if $f_1(n) \prec g_1(n)$ and $f_2(n) \prec g_2(n)$, then $f_1(n) + f_2(n) \prec g_1(n) + g_2(n)$.

Solution. The thesis holds if all the functions above are positive. But if it is not so, then it might be that $f_2(n)$ and $g_2(n)$ remove the components of $f_1(n)$ and $g_1(n)$ which made $f_1(n) \prec g_1(n)$! As an immediate example, take $f_1(n) = n^2 + n$, $f_2(n) = -n^2$, $g_1(n) = n^3 + n$, and $g_2(n) = -n^3$.

Exercise 9.2

Which function grows faster:

1. $n^{\ln n}$ or $(\ln n)^n$?

2. $n^{\ln \ln \ln n}$ or $(\ln n)!$?

3. $(n!)!$ or $((n-1)!)!(n-1)!^{n!}$?

4. $F^2_{\lceil H_n \rceil}$ or H_{F_n}?

Solution. Recall that, if $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = +\infty$, then $f(n) \prec g(n)$ if and only if $\lim_{n\to\infty} (\ln f(n) - \ln g(n)) = -\infty$. As a corollary, if $f(n)$ and $g(n)$ are both positive and $\ln f(n) \prec \ln g(n)$, then $f(n) \prec g(n)$. The vice versa is not true: $\lim_{n\to\infty} \dfrac{n!}{n^n} = 0$ but $\lim_{n\to\infty} \dfrac{\ln n!}{n \ln n} = 1$ by Stirling's approximation.

1. We have $\ln \left(n^{\ln n}\right) = (\ln n)^2$ and $\ln ((\ln n)^n) = n \ln \ln n$: as $(\ln n)^2 \prec n \ln \ln n$, it is $n^{\ln n} \prec (\ln n)^n$.

2. If we switch to natural logarithms, then on the one hand, $\ln n^{\ln\ln\ln n} = \ln n \ln\ln\ln n$, and on the other hand, $\ln(\ln n)!) \asymp \ln n \ln\ln n$: as clearly $\lim_{n\to\infty} \ln n \ln\ln\ln n - \ln n \ln\ln n = -\infty$, switching back to exponentials yields $n^{\ln\ln\ln n} \prec (\ln n)!$.

3. If we switch to logarithms and use Stirling's approximation, on the one hand,
$$\ln((n!)!) \asymp n! \ln n! \asymp n! \cdot n \ln n \,,$$
and on the other hand,
$$\begin{aligned} \ln(((n-1)!)!(n-1)!^{n!}) &= (n-1)! \ln(n-1)! - (n-1)! \\ &\quad + n! \ln(n-1)! \\ &\quad + O(1/(n-1)!) \\ &= (n-1)!\,((n-1)\ln(n-1) - (n-1) + O(1/n)) \\ &\quad + n!\,((n-1)\ln(n-1) - (n-1) + O(1/n)) \\ &\quad + O(1/(n-1)!) \\ &\asymp (n-1)! \cdot (n-1)\ln(n-1)\,. \end{aligned}$$
Then,
$$\begin{aligned} & \ln((n-1)!)!(n-1)!^{n!} - \ln(n!)! \\ \asymp\ & (n-1)! \cdot (n-1)\ln(n-1) - n! \cdot n \ln n \\ =\ & (n-1)!((n-1)\ln(n-1) - n^2 \ln n) \\ \to\ & -\infty \ \text{ for } n \to \infty \,, \end{aligned}$$
whence $((n-1)!)!(n-1)!^{n!} \prec (n!)!$.

4. We know that $F_n \asymp \phi^n$ and $H_n \asymp \ln n$. Then, on the one hand,
$$F^2_{\lceil H_n \rceil} \asymp \phi^{2\ln n} = e^{2\ln n \ln\phi} = n^{2\ln\phi}\,;$$
and on the other hand,
$$H_{F_n} \asymp \ln \phi^n = n \ln \phi \asymp n \,.$$
But $\phi^2 = \phi + 1 = 2.618\ldots < 2.718\ldots = e$: therefore, $2\ln\phi < 1$, and $F^2_{\lceil H_n \rceil} \prec H_{F_n}$.

Exercise 9.3

What's wrong with the following argument? "Since $n = O(n)$ and $2n = O(n)$ and so on, we have $\sum_{k=1}^n kn = \sum_{k=1}^n O(n) = O(n^2)$."

Solution. The conclusion is false, because:

$$\sum_{k=1}^{n} kn = n \cdot \sum_{k=1}^{n} k = n \cdot \frac{n(n+1)}{2} = O(n^3)\,,$$

so something must have gone wrong. What happened is that the functions kn are $O(n)$ as functions of n: not as functions of k. In addition, the multiplicative constants hidden in the O-notation are different as k varies. The kn are actually functions of *two* variables, k and n: not of the single variable n. However, as the sum is from k from 1 to n, it is $k = O(n)$, so the correct argument is:

$$\sum_{k=1}^{n} kn = \sum_{k=1}^{n} O(n) \cdot n = \sum_{k=1}^{n} O(n^2) = O(n^3)\,.$$

Exercise 9.7

Estimate $\sum_{k \geqslant 0} e^{-k/n}$ with absolute error $O(n^{-1})$.

Solution. Since for $n \geqslant 1$ it is $e^{-k/n} = \left(e^{-1/n}\right)^k$ and $e^{-1/n} < 1$, we have:

$$\begin{aligned}
\sum_{k \geqslant 0} e^{-k/n} &= \frac{1}{1 - e^{-1/n}} \\
&= n \cdot \frac{-1/n}{e^{-1/n} - 1} \\
&= n \cdot \sum_{k \geqslant 0} \frac{B_k}{k!} \left(-\frac{1}{n}\right)^k \\
&= n \cdot \left(B_0 - \frac{B_1}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= n + \frac{1}{2} + O\left(\frac{1}{n}\right)\,.
\end{aligned}$$

Exercise 9.8

Give an example of functions $f(n)$ and $g(n)$ such that none of the three relations $f(n) \prec g(n)$, $g(n) \prec f(n)$, $f(n) \asymp g(n)$ is valid, although $f(n)$ and $g(n)$ both increase monotonically to ∞.

Solution. The idea is to find $f(n)$ and $g(n)$ such that $\liminf_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$ and $\limsup_{n\to\infty} \dfrac{f(n)}{g(n)} = +\infty$. Let's try:

$$\begin{aligned} f(n) &= (\lfloor n/2 \rfloor !)^2 + n\,, \\ g(n) &= (\lceil n/2 \rceil - 1)!\,\lceil n/2 \rceil! + n\,. \end{aligned}$$

Note how the summand n makes the functions strictly increasing. If $n = 2m$ is even, then:

$$\frac{f(n)}{g(n)} = \frac{(m!)^2 + 2m}{(m-1)!m! + 2m} = m \cdot \Theta(1)\,,$$

so $f(n) \not\prec g(n)$; if $n = 2m + 1$ is odd, then:

$$\frac{f(n)}{g(n)} = \frac{(m!)^2 + 2m + 1}{m!(m+1)! + 2m + 1} = \frac{1}{m+1} \cdot \Theta(1)\,,$$

so $g(n) \not\prec f(n)$ either. Since both functions are ultimately positive and the ratio $\dfrac{f(n)}{g(n)} = \dfrac{|f(n)|}{|g(n)|}$ becomes both arbitrarily small and arbitrarily large, it cannot be $f(n) \asymp g(n)$ either.

Exercise 9.11

Prove or disprove: $O(x + y)^2 = O(x^2) + O(y^2)$.

Solution. This time, we have a complication in that the big-O notation depends on two variables, not one. However, it must be either $|x| \leqslant |y|$ or $|y| \leqslant |x|$, and since the roles of x and y are symmetric, we can consider only one of the two cases.

So suppose $|x| \leqslant |y|$. Then $|x + y|^2 \leqslant 4|y|^2$, so surely $(x + y)^2 = O(y^2) = O(x^2) + O(y^2)$. (Recall that we use $O(f(x)) = O(g(x))$ as a shortcut for

$O(f(x)) \subseteq O(g(x))$.) Then:

$$\begin{aligned} O(x+y)^2 &= O((x+y)^2) \\ &= O((O(x^2)+O(y^2)) \\ &= O(O(x^2))+O(O(y^2)) \\ &= O(x^2)+O(y^2)\,. \end{aligned}$$

Exercise 9.12

Prove that

$$1+\frac{2}{n}+O(n^{-2}) = \left(1+\frac{2}{n}\right)(1+O(n^{-2}))\,,$$

as $n \to \infty$.

Solution. The deduction is not immediate: if we isolate a factor $1+\dfrac{2}{n}$, all we can conclude is that

$$1+\frac{2}{n}+O(n^{-2}) = \left(1+\frac{2}{n}\right)\cdot\left(1+\frac{1}{1+\frac{2}{n}}\cdot O(n^{-2})\right)$$

However, for $n \geqslant 1$ it is $1+\dfrac{2}{n} > 1$, so clearly $(1+2/n)^{-1} = O(1)$: we can then conclude

$$\begin{aligned} 1+\frac{2}{n}+O(n^{-2}) &= \left(1+\frac{2}{n}\right)\cdot\left(1+O(1)\cdot O(n^{-2})\right) \\ &= \left(1+\frac{2}{n}\right)\cdot\left(1+O(1\cdot n^{-2})\right) \\ &= \left(1+\frac{2}{n}\right)\cdot(1+O(n^{-2}))\,. \end{aligned}$$

Exercise 9.13

Evaluate $(n+2+O(1/n))^n$ with a relative error $O(1/n)$.

Solution. This time, we are looking for a *relative* error, that is, some expression $h(n)$ such that $(n+2+O(1/n))^n = h(n)\cdot(1+O(1/n))$. The first idea is to take out a factor n^n:

$$(n+2+O(1/n))^n = n^n\cdot(1+2/n+O(1/n^2))^n\,.$$

Does this help? A bit, because $1 + 2/n + O(1/n^2)$ is an approximation for $e^{2/n+O(1/n^2)}$, so we can substitute:

$$\begin{aligned} n^n \cdot (1 + 2/n + O(1/n^2))^n &= n^n \cdot e^{(2/n+O(1/n^2))\cdot n} \\ &= n^n \cdot e^{2+O(1/n)} \\ &= n^n \cdot (e^2 + O(1/n)) \\ &= e^2 n^n (1 + O(1/n)) \end{aligned}$$

because clearly $e^2 \cdot O(1/n) = O(1/n)$. Alternatively: $n^n \cdot (1+2/n+O(1/n^2))^n = e^2 n^n + O(n^{n-1})$.

Exercise 9.14

Show that $(n + \alpha)^{n+\beta} = n^{n+\beta} e^{\alpha} \left(1 + \alpha \frac{\beta - \alpha/2}{n} + O\left(\frac{1}{n^2}\right)\right)$.

Solution. As $(n + \alpha)^{n+\beta} = n^{n+\beta} \left(1 + \frac{\alpha}{n}\right)^{n+\beta}$, we only need to prove:

$$\left(1 + \frac{\alpha}{n}\right)^{n+\beta} = e^{\alpha} \left(1 + \alpha \frac{\beta - \alpha/2}{n} + O\left(\frac{1}{n^2}\right)\right) .$$

But

$$\left(1 + \frac{\alpha}{n}\right)^{n+\beta} = e^{(n+\beta)\ln\left(1+\frac{\alpha}{n}\right)} ,$$

and we can work on the argument of the exponential:

$$\begin{aligned} (n + \beta) \ln\left(1 + \frac{\alpha}{n}\right) &= (n + \beta) \cdot \left(\frac{\alpha}{n} - \frac{\alpha^2}{2n^2} + O\left(\frac{1}{n^3}\right)\right) \\ &= \alpha + \frac{1}{n} \cdot \left(\beta\alpha - \frac{\alpha^2}{2}\right) + O\left(\frac{1}{n^2}\right) \\ &= \alpha + \alpha \frac{\beta - \alpha/2}{n} + O\left(\frac{1}{n^2}\right) . \end{aligned}$$

Then, as $e^{f(n)} = 1 + f(n) + O((f(n))^2)$ when $f(n) = O(1)$:

$$\begin{aligned} e^{(n+\beta)\ln\left(1+\frac{\alpha}{n}\right)} &= e^{\alpha} \cdot e^{\alpha \frac{\beta-\alpha/2}{n} + O\left(\frac{1}{n^2}\right)} \\ &= e^{\alpha} \left(1 + \alpha \frac{\beta - \alpha/2}{n} + O\left(\frac{1}{n^2}\right)\right) , \end{aligned}$$

which concludes our proof.