ITT9132 Concrete Mathematics
Exercises from Week 16 – 15 May 2019

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Final exam of 3 January 2017

Exercise 1

(12 points) Solve the recurrence:

$$\begin{aligned} g_0 &= 0\,; & g_1 &= 2\,; \\ g_n &= \frac{5}{2}\,g_{n-1} - g_{n-2} & \forall n &\geqslant 2\,. \end{aligned} \tag{1}$$

Solution. The recurrence (1) is easily solved with generating functions via the Rational Expansion Theorem. Let us follow the method step by step:

1. We rewrite (1) so that it holds for every $n \in \mathbb{Z}$, with the convention that $g_n = 0$ if $n < 0$. We have to check the cases $n = 0$ and $n = 1$:

   - For $n = 0$ we have $g_0 = 0$ and $\frac{5}{2}g_{-1} - g_{-2} = 0$. Thus, we need no correction summand.
   - For $n = 1$ we have $g_1 = 2$ but $\frac{5}{2}g_0 - g_{-1} = 0$. Thus, we need to add a correction summand 2.

   The equation (1) rewritten for arbitrary $n \in \mathbb{Z}$ becomes:

   $$g_n = \frac{5}{2}g_{n-1} - g_{n-2} + 2\,[n = 1] \quad \forall n \in \mathbb{Z}\,. \tag{2}$$

2. Let $G(z) = \sum_n g_n z^n$ be the generating function of the sequence $\langle g_n \rangle$. By multiplying (2) by z^n for every $n \in \mathbb{Z}$ and summing over n we obtain:

$$\begin{aligned} G(z) &= \sum_n g_n z^n \\ &= \frac{5}{2} \sum_n g_{n-1} z_n - \sum_n g_{n-2} z^n + 2 \sum_n [n=1]\, z^n \\ &= \frac{5}{2} \sum_n g_n z^{n+1} - \sum_n g_n z^{n+2} + 2z \\ &= \frac{5}{2}\, z\, G(z) - z^2\, G(z) + 2z\,. \end{aligned}$$

3. By solving the above with respect to $G(z)$ we get

$$G(z) \cdot \left(1 - \frac{5}{2} z + z^2\right) = 2z\,,$$

which yields

$$G(z) = \frac{2z}{1 - \frac{5}{2} z + z^2}\,. \tag{3}$$

4. Equation (3) has the form $G(z) = P(z)/Q(z)$ where $P(z) = 2z$ and $Q(z) = 1 - \frac{5}{2} z + z^2 = (1-2z)(1-z/2)$. We can then apply the Rational Expansion Theorem with $\rho_1 = 2$, $\rho_2 = 1/2$, and $d_1 = d_2 = 1$. As $Q'(z) = 2z - \frac{5}{2}$, we find

$$a_1 = \frac{(-2) \cdot (2 \cdot 1/2)}{2 \cdot 1/2 - 5/2} = \frac{-2}{-3/2} = \frac{4}{3}$$

and

$$a_2 = \frac{(-1/2) \cdot (2 \cdot 2)}{2 \cdot 2 - 5/2} = \frac{-2}{3/2} = -\frac{4}{3}\,.$$

Alternatively:

$$a_1 = \frac{2 \cdot 1/2}{0!(1 - (1/2)/2)} = \frac{1}{1 - 1/4} = \frac{4}{3} \text{ and } a_2 = \frac{2 \cdot 1/(1/2)}{0!(1 - 2 \cdot 1/(1/2))} = \frac{4}{1-4} = -\frac{4}{3}\,.$$

We can thus conclude:

$$g_n = \frac{4}{3}\left(2^n - \frac{1}{2^n}\right)\,.$$

Exercise 2

(10 points) For $n \in \mathbb{N}$ and $r, s \in \mathbb{R}$ compute

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{r+k}{k} \binom{s}{n-k} \,.$$

Solution. The sequence $\langle S_n \rangle$ is the convolution of the sequences $\left\langle (-1)^n \binom{r+n}{n} \right\rangle$ and $\left\langle \binom{s}{n} \right\rangle$. We know that $\sum_{n \geqslant 0} \binom{r+n}{n} z^n = 1/(1-z)^{r+1}$ and $\sum_{n \geqslant 0} \binom{s}{n} = (1+z)^s$: by replacing z with $-z$ in the first power series, we obtain

$$\sum_{n \geqslant 0} (-1)^n \binom{r+n}{n} z^n = \frac{1}{(1+z)^{r+1}} \,.$$

(Alternatively, since $c^{\underline{n}} = (-1)^n (-c)^{\overline{n}} = (-1)^n (n-1-c)^{\underline{n}}$ for every $c \in \mathbb{R}$, by putting $c = r+n$ we obtain $(-1)^n \binom{r+n}{n} = \binom{-r-1}{n}$, which yields the same generating function.) Then the generating function of $\langle S_n \rangle$ is

$$S(z) = \frac{(1+z)^s}{(1+z)^{r+1}} = (1+z)^{s-r-1} \ :$$

it follows immediately that

$$\sum_{k=0}^{n} (-1)^k \binom{r+k}{k} \binom{s}{n-k} = \binom{s-r-1}{n} \,.$$

Exercise 3

(8 points) Determine for which integer values of n the number $n^{13} - 2n^7 + n$ is divisible by 98.

Solution. As $98 = 2 \cdot 7^2$ as a product of powers of primes, we must show that $n^{13} - 2n^7 + n$ is divisible by both 2 and 49. One part is easy: the second summand is even, and the other two are either both even or both odd, so the sum is even. For the other part, we factor the polynomial and obtain:

$$n^{13} - 2n^7 + n = n \cdot (n^{12} - 2n^6 + 1) = n \cdot (n^6 - 1)^2 \,.$$

If n is not a multiple of 7, then $n^6 - 1$ is by Fermat's little theorem, and as there are two such factors, $n^{13} - 2n^7 + n$ is indeed divisible by 49. If n is a multiple of 7, however, then $n^6 - 1$ is not, and since we only have one factor n, it must be n that is divisible by 49.

In conclusion, $n^{13} - 2n^7 + n$ is divisible by 98 if and only if n is either divisible by 49, or not divisible by 7.

Final exam of 17 January 2017

Exercise 1

(12 points) Solve the recurrence:

$$\begin{aligned} g_0 &= 1\,; & g_1 &= 3\,; \\ g_n &= 4g_{n-1} - 4g_{n-2} & \forall n &\geqslant 2\,. \end{aligned} \tag{4}$$

Solution. The recurrence (4) is easily solved with generating functions via the Rational Expansion Theorem. Let us follow the method step by step:

1. We must rewrite (4) so that it holds for every $n \in \mathbb{Z}$, with the convention that $g_n = 0$ if $n < 0$. We need to check the initial conditions:

   - For $n = 0$ it is $g_0 = 1$ but $4g_{-1} - 4g_{-2} = 0$: we thus need a correction summand 1.
   - For $n = 1$ it is $g_1 = 3$ but $4g_0 - 4g_{-1} = 4$: we thus need a correction summand -1.

   The recurrence (4) for arbitrary $n \in \mathbb{Z}$ is thus:

   $$g_n = 4g_{n-1} - 4g_{n-2} + [n = 0] - [n = 1]\ .$$

2. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. By multiplying the recurrence by z^n for every $n \in \mathbb{Z}$ and summing over n we obtain:

   $$\begin{aligned} G(z) &= \sum_n g_n z^n \\ &= 4\sum_n g_{n-1} z^n - 4\sum_n g_{n-2} z^n + \sum_n [n = 0]\, z^n - \sum_n [n = 1]\, z^n \\ &= 4\sum_n g_n z^{n+1} - 4\sum_n g_n z^{n+2} + 1 - z \\ &= 4zG(z) - 4z^2 G(z) + 1 - z\,. \end{aligned}$$

3. By solving the above with respect to $G(z)$ we get

   $$G(z) \cdot \left(1 - 4z + 4z^2\right) = 1 - z\,,$$

which yields

$$G(z) = \frac{1-z}{1-4z+4z^2}\,.$$

4. The function $G(z)$ has the form $G(z) = P(z)/Q(z)$ where $P(z) = 1-z$ and $Q(z) = 1-4z+4z^2 = (1-2z)^2$. Then the solution of the recurrence is $(an+b)\cdot 2^n$ for suitable a and b. To find such numbers, we use the Rational Expansion Theorem: in our case, $\rho = 2$ and $d = 2$, so:

$$a = \frac{(-2)^2 \cdot P(1/2) \cdot 2}{Q''(1/2)} = \frac{4\cdot(1/2)\cdot 2}{8} = \frac{1}{2}\,,$$

or alternatively,

$$a = \frac{1-1/2}{1!\cdot(\text{empty product})} = \frac{1}{2}\,.$$

To find b, we compare the initial condition $g_0 = 1$ with the value $(a\cdot 0+b)\cdot 2^0$: which yields $b = 1$. In conclusion,

$$g_n = \left(\frac{n}{2}+1\right)\cdot 2^n\,.$$

Exercise 2

(10 points) For $n, r, s \geqslant 0$ all integers compute

$$S_n = \sum_{k=0}^{n} \binom{k}{r}\binom{n-k}{s}\,.$$

Solution. The sequence $\langle S_n\rangle$ is the convolution of the sequences $\left\langle \binom{n}{r} \right\rangle$ and $\left\langle \binom{n}{s} \right\rangle$. We know that $\sum_{n\geqslant 0}\binom{n}{r} = \frac{z^r}{(1-z)^{r+1}}$ and $\sum_{n\geqslant 0}\binom{n}{s} = \frac{z^s}{(1-z)^{s+1}}$: then the generating function of $\langle S_n\rangle$ is

$$S(z) = \frac{z^{r+s}}{(1-z)^{r+s+2}}\,.$$

This writing is annoying, because the right-hand side does not have the convenient form $\frac{z^m}{(1-z)^{m+1}}$: which it would have if the exponent at the numerator was $r+s+1$ instead of $r+s$. But as $r+s \geqslant 0$, the constant coefficient of $\dfrac{z^{r+s+1}}{(1-z)^{r+s+2}} = \sum_{n\geqslant 0}\binom{n}{r+s+1}z^n$ is $\binom{0}{r+s+1} = 0$: by applying the formula $\dfrac{G(z)-g_0}{z} = \sum_{n\geqslant 0} g_{n+1}z^n$, we get

$$S(z) = \frac{1}{z}\cdot\left(\frac{z^{r+s+1}}{(1-z)^{r+s+2}} - 0\right) = \sum_{n\geqslant 0}\binom{n+1}{r+s+1}z^n\,.$$

By comparison, we finally find:

$$\sum_{k=0}^{n}\binom{k}{r}\binom{n-k}{s} = \binom{n+1}{r+s+1}\,.$$

Exercise 3

(8 points) Determine the values of $n \geqslant 0$ such that $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 250.

Solution. As $250 = 2 \cdot 5^3$ as a product of powers of primes, we must show that $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by both 2 and 125. One part is easy: there are four summands, which are either all even or all odd, so the sum is even. For the other part, we factor the polynomial and obtain:

$$n^{14} - 3n^{10} + 3n^6 - n^2 = n^2 \cdot (n^{12} - 3n^8 + 3n^4 - 1) = n^2 \cdot (n^4 - 1)^3\,.$$

If n is not a multiple of 5, then $n^4 - 1$ is by Fermat's little theorem, and as there are three such factors, $n^{14} - 3n^{10} + 3n^6 - n^2$ is indeed divisible by 125. If n is a multiple of 5, however, then $n^4 - 1$ is not, and the contributions to divisibility by 125 must come all from n: as there are two factors n in $n^{14} - 3n^{10} + 3n^6 - n^2$, if n is divisible by 5 but not by 25, then $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 25 but not by 125; while if n is divisible by 25, then $n^4(n^4 - 1)^3$ is divisible by 625, thus also by 125.

In conclusion, $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 250 if and only if n is either divisible by 25, or not divisible by 5.

Exercises from Chapter 9

Exercise 9.15

Give an asymptotic formula for the "middle" trinomial coefficient $\binom{3n}{n,n,n}$, correct to relative error $O(n^{-3})$.

Solution. Recall the definition of trinomial coefficient: if $k_1 + k_2 + k_3 = n$, then $\binom{n}{k_1,k_2,k_3} = \dfrac{n!}{k_1!k_2!k_3!}$. Having to do with factorials, we resort to Stirling's approximation as given in (9.91):

$$\ln n! \;=\; \left(n + \frac{1}{2}\right) \ln n - n + \ln \sqrt{2\pi} + \frac{1}{12n} + O\left(\frac{1}{n^3}\right) .$$

So:

$$\begin{aligned}
\ln \binom{3n}{n,n,n} &= \ln(3n)! - \ln(n!)^3 \\
&= \left(3n + \frac{1}{2}\right)(\ln 3 + \ln n) - 3n + \ln \sqrt{2\pi} + \frac{1}{12 \cdot 3n} \\
&\quad -3\left(\left(n + \frac{1}{2}\right) \ln n - n + \ln \sqrt{2\pi} + \frac{1}{12n}\right) + O\left(\frac{1}{n^3}\right) \\
&= \left(3n + \frac{1}{2}\right) \ln 3 - \ln n - \ln 2\pi + \frac{1}{12}\left(\frac{1}{3} - 3\right) \cdot \frac{1}{n} + O\left(\frac{1}{n^3}\right) \\
&= \left(3n + \frac{1}{2}\right) \ln 3 - \ln n - \ln 2\pi - \frac{2}{9n} + O\left(\frac{1}{n^3}\right) .
\end{aligned}$$

By switching to exponentials,

$$\binom{3n}{n,n,n} = \frac{3^{3n+1/2}}{2\pi n} \cdot e^{-2/9n+O(1/n^3)} :$$

and we are almost done, but not fully, because we need a relative error $O(1/n^3)$, so we also need to consider *the power series development of e^z up to the quadratic term.* Which we do:

$$\begin{aligned}
e^{-2/9n+O(1/n^3)} &= 1 - \frac{2}{9n} + O\left(\frac{1}{n^3}\right) + \frac{1}{2} \cdot \left(-\frac{2}{9n} + O\left(\frac{1}{n^3}\right)\right)^2 + O\left(\frac{1}{n^3}\right) \\
&= 1 + \left(-\frac{2}{9} + O\left(\frac{1}{n^3}\right)\right) + \left(\frac{2}{81n^2} + O\left(\frac{1}{n^4}\right)\right) + O\left(\frac{1}{n^3}\right) \\
&= 1 - \frac{2}{9} + \frac{2}{81n^2} + O\left(\frac{1}{n^3}\right) .
\end{aligned}$$

We can conclude:

$$\binom{3n}{n,n,n} = \frac{3^{3n+1/2}}{2\pi n} \cdot \left(1 - \frac{2}{9n} + \frac{2}{81n^2} + O\left(\frac{1}{n^3}\right)\right) .$$