ITT9132 Concrete Mathematics
Final exam

First date, 22 May 2019

Full name: Code:

1. Take note of the code near your full name: it will be used to display the results.
2. Write your solutions under the corresponding exercise or question. For Exercises 1, 2 and 3, explain your reasoning.
3. You may use any formula seen in classroom or appearing in the self-evaluation tests.
4. You may use the additional paper to draft your answers. However, only what is written in the exercises' pages will be evaluated.
5. Partially completed exercises may receive a fraction of the total score.
6. Only handwritten notes are allowed.
7. Electronic devices, including mobile phones must be turned off. Using a pocket or tabletop calculator is allowed as the only exception.
8. It is permitted to leave the room once, for a maximum of 5 minutes, one at a time, handing the assignment to the instructor, who will give it back on return.

Exercise 1 (12 points)

Solve the recurrence

$$g_n = 5g_{n-1} - 6g_{n-2} \quad \text{for every } n \geqslant 2$$

with the initial conditions $g_0 = 0, g_1 = 3$.

Solution. The recurrence is a second-order homogenoeus linear recurrence, which is solved easily with generating functions and the Rational Expansion Theorem. Let us follow step by step:

1. We rewrite the recurrence so that it holds for every n integer, with the convention that $g_n = 0$ if $n < 0$. We expect some correction terms in correspondence of the initial conditions, so our recurrence will take the form:

   $$g_n = 5g_{n-1} - 6g_{n-2} + a_0\,[n=0] + a_1\,[n=1] \quad \text{for every } n \in \mathbb{Z}$$

   for suitable a_0 and a_1. Now:

   - For $n = 0$ it is $g_0 = 0$ and $5g_{-1} - 6g_{-2} = 0$. Hence, $a_0 = 0$.
   - For $n = 1$ it is $g_1 = 3$ but $5g_0 - 6g_{-1} = 0$. Hence, $a_1 = 3$.

   Summarizing:

   $$g_n = 5g_{n-1} - 6g_{n-2} + 3\,[n=1] \quad \text{for every } n \in \mathbb{Z}$$

2. By multiplying by z^n and summing over $n \in \mathbb{Z}$ we obtain:

   $$\sum_n g_n z^n = 5\sum_n g_{n-1} z^n - 6\sum_n g_{n-1} z^n + 3\sum_n [n=1]\, z^n\,.$$

   Calling $G(z)$ the left-hand side, the above can be rewritten:

   $$G(z) = 5zG(z) - 6z^2G(z) + 3z\,;$$

   carrying all the terms with $G(z)$ to the left-hand side we obtain:

   $$G(z) \cdot (1 - 5z + 6z^2) = 3z\,,$$

   which gives the following expression for the generating function:

   $$G(z) = \frac{3z}{1 - 5z + 6z^2}\,.$$

3. Let $P(z)$ and $Q(z)$ be the numerator and denominator in the expression of $G(z)$. The reflected polynomial of the denominator is $Q^R(z) = z^2 - 5z + 6$, which has the roots $z = 2$ and $z = 3$: consequently, $Q(z) = (1-2z)(1-3z)$ is the product of two factors of the first degree. The Rational Expansion Theorem gives $\rho_1 = 2$, $\rho_2 = 3$ and $d_1 = d_2 = 1$, so:

$$g_n = a_1 \cdot 2^n + a_2 \cdot 3^n \quad \text{for every } n \geqslant 0\,,$$

where the coefficients a_1 and a_2 are computed as:

$$\begin{aligned} a_1 &= \frac{3 \cdot 1/2}{1 - 3 \cdot 1/2} = \frac{3/2}{-1/2} = -3\,; \\ a_2 &= \frac{3 \cdot 1/3}{1 - 2 \cdot 1/3} = \frac{1}{1/3} = 3\,. \end{aligned}$$

4. We can now conclude:

$$g_n = 3 \cdot (3^n - 2^n) \quad \text{for every } n \geqslant 0\,.$$

Exercise 2 (10 points)

For $n, m \geqslant 0$ integers compute:

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{k+m}{m} \binom{n-k+m}{m}\,.$$

Solution. The sequence $\langle S_n \rangle$ is the convolution of the sequences $\left\langle (-1)^n \binom{n+m}{m} \right\rangle$ and $\left\langle \binom{n+m}{m} \right\rangle$. The generating function of the latter is

$$\sum_{n \geqslant 0} \binom{n+m}{m} z^n = \sum_{n \geqslant 0} \binom{n+m}{n} z^n = \frac{1}{(1-z)^{m+1}}\,,$$

because m is integer; for the other, we observe that multiplying the nth term by $(-1)^n$ corresponds to evaluating the generating function in $-z$ instead of z, so $\sum_{n \geqslant 0} (-1)^n \binom{n+m}{m} z^n = \dfrac{1}{(1+z)^{m+1}}$. Then the generating function of the sequence $\langle S_n \rangle$ is:

$$\begin{aligned} S(z) &= \frac{1}{(1+z)^{m+1}} \cdot \frac{1}{(1-z)^{m+1}} \\ &= \frac{1}{(1-z^2)^{m+1}} \\ &= \sum_{n \geqslant 0} \binom{n+m}{m} z^{2n} \\ &= \sum_{n \geqslant 0} \binom{\lfloor n/2 \rfloor + m}{m} [n \text{ is even}]\, z^n \,. \end{aligned}$$

By comparing the coefficients, we obtain:

$$S_n = \binom{\lfloor n/2 \rfloor + m}{m} [n \text{ is even}] \,.$$

Exercise 3 (8 points)

Determine for which integer values $n \geqslant 0$ the number $n^{21} - 2n^{11} + n$ is divisible by 242.

Solution. As $242 = 2 \cdot 11^2$ as a product of primes, $n^{21} - 2n^{11} + n$ is divisible by 242 if and only if it is even and divisible by 121. The first part is easy: of the three summands, the middle one is even, and the other two are either both even or both odd, so the sum is even. Now:

$$n^{21} - 2n^{11} + n = n \cdot (n^{20} - 2n^{10} + 1) = n \cdot (n^{10} - 1)^2 \,.$$

If n is not divisible by 11, then $n^{10} - 1$ is by Fermat's little theorem, and as there are two such factors, $n^{21} - 2n^{11} + n$ is divisible by 121; if n is divisible by 11, then $n^{10} - 1$ is not, so it is n which must be divisible by 121. In conclusion, $n^{21} - 2n^{11} + n$ is divisible by 242 if and only if n is either not divisible by 11, or divisible by 121.

Exercise 4 (1 point each, 20 points total)

1. Twenty people are sitting in circle and every second one is eliminated. Who remains last?

   The ninth one: $20 = 16 + 4$ *and* $2 \cdot 4 + 1 = 9$.

2. Describe the perturbation method.

   Given a sum of the form $S_n = \sum_{0 \leqslant k \leqslant n} a_k$*, we rewrite* S_{n+1} *as* $S_n + a_{n+1}$ *on one side and* $a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k$ *on the other, and solve with respect to* S_n*.*

3. Write a function $u(x)$ such that $\Delta u(x) = \left(\frac{5}{2}\right)^x$.

   $u(x) = \frac{2}{3} \cdot \left(\frac{5}{2}\right)^x$. *Then,* $\Delta u(x) = \frac{2}{3} \cdot \left(\frac{5}{2}\right)^x \cdot \left(\frac{5}{2} - 1\right) = \left(\frac{5}{2}\right)^x$.

4. Let $\sum_{n \geqslant 0} a_n$ be an infinite sum and let p be a permutation of $\mathbb{N}$. What is a sufficient condition to have $\sum_{n \geqslant 0} a_n = \sum_{n \geqslant 0} a_{p(n)}$?

   That $\sum_{n \geqslant 0} a_n$ *converges absolutely, that is,* $\sum_{n \geqslant 0} |a_n| < \infty$.

5. True or false: for every $x > 0$, $\left\lfloor \sqrt{x/10} \right\rfloor = \left\lfloor \sqrt{\lfloor x \rfloor /10} \right\rfloor$.

   True: the function $f(x) = \sqrt{x/10}$ *is continuous and strictly increasing on the positive reals and if* $f(x) = k$ *is integer, so is* $x = 10k^2$.

6. How many integers $1 \leqslant k \leqslant n$ are in the union of the spectra of $\alpha = \sqrt{5}$ and $\beta = \frac{5 + \sqrt{5}}{4}$?

   n*. As* α *and* β *are both irrational and* $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\sqrt{5}} + \frac{4}{5 + \sqrt{5}} = 1$*, the spectra of* α *and* β *form a partition of the positive integers.*

7. What is a Fermat pseudoprime for base b?

   A composite number n *such that* $b^{n-1} \equiv 1 \pmod{n}$.

8. Is $105^{120} - 1$ divisible by 41?

   Yes: as 41 is prime and $105 = 3 \cdot 5 \cdot 7$, $105^{120} - 1 = (105^{40} - 1) \cdot (105^{80} + 105^{40} + 1)$ *is divisible by 41 by Fermat's little theorem.*

9. Let $n = pq$ be the product of two distinct primes. What is the value $\phi(n)$ of Euler's totient function ϕ on n?

   $(p-1)(q-1)$*. As* p *and* q *are prime,* $\phi(p) = p - 1$ *and* $\phi(q) = q - 1$*, and as* p *and* q *are distinct,* $p \perp q$*, and* $\phi(pq) = \phi(p) \cdot \phi(q)$.

10. True or false: for every r complex and k nonnegative integer, $(r-k)\binom{r}{k} = r\binom{r-1}{k}$.

    True: the identity is easily seen to hold if r is an arbitrary positive integer, and as both sides are polynomials of degree at most $k+1$ in the variable r, it holds for every r complex.

11. Let $G(z)$ be a power series with center 0 and convergence radius 1. Can it be that $G(z)$ converges at every z such that $|z| = 1$?

    Yes: for example, $G(z) = \sum_{n \geqslant 1} \frac{z^n}{n^2}$ converges totally on the closed unit disk.

12. Write the recurrence equation for the Stirling numbers of the second kind.

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}.$$

13. How many ways are there to arrange 5 objects into 2 nonempty cycles?

    $\left[{5 \atop 2} \right] = 4! H_4 = 24 + 12 + 8 + 6 = 50.$

14. By how much can a stack of cards hang out of a table without toppling?

    As much as we want (provided we have enough many cards). More precisely, a stack of n cards can hang out by H_n "half cards". See Lecture 12.

15. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. Given $m \geqslant 1$ integer, what is the generating function of the sequence $\langle g_{m+n} \rangle$?

    $\dfrac{G(z) - g_0 - \ldots - g_{m-1} z^{m-1}}{z^m}.$

16. What is the generating function of the sequence of the natural numbers?

    $\sum_{n \geqslant 0} n z^n = \dfrac{z}{(1-z)^2}.$

17. Write the generating function of the sequence $\left\langle \binom{c}{n} \right\rangle$, where c is a complex number.

    $\sum_{n \geqslant 0} \binom{c}{n} z^n = (1+z)^c.$

18. Name a sequence $\langle g_n \rangle$ which does not have an analytic generating function, but is such that $\langle g_n/n! \rangle$ has.

    There are many: the sequence of the Bernoulli numbers, the sequence of the factorials, the sequence $\langle n^m \cdot n! \rangle$ *where* m *is a fixed integer, etc.*

19. True or false: for any $f(n)$ and $g(n)$, $O(f(n)) + O(g(n)) = O(f(n) + g(n))$.

    False: if $f(n) = n+1$ *and* $g(n) = -n$, *then* $O(f(n)) + O(g(n)) = O(n)$ *but* $O(f(n) + g(n)) = O(1)$.

20. True or false: $\ln\left(1 + \dfrac{1}{\ln n}\right) = \dfrac{1}{\ln n} - \dfrac{1}{2(\ln n)^2} + O\left(\left(\dfrac{1}{\ln n}\right)^3\right)$.

    True, because $\ln(1+z) = z - \dfrac{z^2}{2} + \sum_{k \geqslant 3} \dfrac{(-1)^{k-1}}{k} z^k$ *and* $\dfrac{1}{\ln n} \prec 1$.

    (Corrects a wrong version where $O(1/\ln n)$ appeared on the left-hand side in place of $1/\ln n$.)