ITT9132 Concrete Mathematics
Final exam

Second date, 5 June 2019

Full name: Code:

1. Take note of the code near your full name: it will be used to display the results.
2. Write your solutions under the corresponding exercise. For Exercises 1, 2 and 3, explaining your reasoning.
3. You may use any formula seen in classroom or appearing in the self-evaluation tests.
4. You may use the additional paper to draft your answers. However, only what is written in the exercises' pages will be evaluated.
5. Partially completed exercises may receive a fraction of the total score.
6. Only handwritten notes are allowed.
7. Electronic devices, including mobile phones must be turned off. Using a pocket or tabletop calculator is allowed as the only exception.
8. It is permitted to leave the room once, for a maximum of 5 minutes, one at a time, handing the assignment to the instructor, who will give it back on return.

Exercise 1 (12 points)

Solve the recurrence:

$$g_n \quad = \quad 2g_{n-1} + 8g_{n-2} \text{ for every } n \geqslant 2$$

with the initial conditions $g_0 = 1, g_1 = 3$.

Solution. The recurrence is a second-order homogenoeus linear recurrence, which is solved easily with generating functions and the Rational Expansion Theorem. Let us follow step by step:

1. We rewrite the recurrence so that it holds for every n integer, with the convention that $g_n = 0$ if $n < 0$. We expect some correction terms in correspondence of the initial conditions, so our recurrence will have the form:

   $$g_n = 2g_{n-1} + 8g_{n-2} + c_0\,[n = 0] + c_1\,[n = 1] \quad \text{for every } n \in \mathbb{Z}$$

   for suitable a_0 and a_1. Now:

   - For $n = 0$ it is $g_0 = 1$ but $2g_{-1} + 8g_{-2} = 0$. Hence, $c_0 = 1$.
   - For $n = 1$ it is $g_1 = 3$ but $2g_0 + 8g_{-1} = 2$. Hence, $c_1 = 3 - 2 = 1$.

   Summarizing:

   $$g_n = 2g_{n-1} + 8g_{n-2} + [n = 0] + [n = 1] \quad \text{for every } n \in \mathbb{Z}$$

2. By multiplying by z^n and summing over $n \in \mathbb{Z}$ we obtain:

   $$\sum_n g_n z^n \quad = \quad 2\sum_n g_{n-1}z^n + 8\sum_n g_{n-1}z^n + \sum_n [n = 0] + \sum_n [n = 1]\, z^n\,.$$

   Calling $G(z)$ the left-hand side, the above can be rewritten:

   $$G(z) \quad = \quad 2zG(z) + 8z^2G(z) + 1 + z\,.$$

3. Carrying all the terms with $G(z)$ to the left-hand side we obtain:

   $$G(z) \cdot (1 - 2z - 8z^2) \quad = \quad 1 + z\,,$$

   which gives the following expression for the generating function:

   $$G(z) = \frac{1 + z}{1 - 2z - 8z^2}\,.$$

4. Let $P(z)$ and $Q(z)$ be the numerator and denominator in the expression of $G(z)$. The reflected polynomial of the denominator is $Q^R(z) = z^2 - 2z - 8$, which has the roots $z = -2$ and $z = 4$: consequently, $Q(z) = (1+2z)(1-4z)$ is the product of two factors of the first degree. The Rational Expansion Theorem gives $\rho_1 = -2$, $\rho_2 = 4$ and $d_1 = d_2 = 1$, so:

$$g_n = a_1 \cdot (-2)^n + a_2 \cdot 4^n \text{ for every } n \geqslant 0 \,,$$

where the coefficients a_1 and a_2 are computed as:

$$\begin{aligned} a_1 &= \frac{1 + 1/(-2)}{1 - 4 \cdot 1/(-2)} = \frac{1/2}{3} = \frac{1}{6}\,; \\ a_2 &= \frac{1 + 1/4}{1 + 2 \cdot 1/4} = \frac{5/4}{3/2} = \frac{5}{6}\,. \end{aligned}$$

We conclude:

$$g_n = \frac{(-2)^n + 5 \cdot 4^n}{6}\,.$$

Exercise 2 (10 points)

Let $r \in \mathbb{R}$. For $m, n \geqslant 0$ integers compute:

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{r+k}{k} \binom{r+1}{n-k-m}\,.$$

Solution. The sequence $\langle S_n \rangle$ is the convolution of the sequences $\left\langle (-1)^n \binom{r+n}{n} \right\rangle$ and $\left\langle \binom{r+1}{n-m} \right\rangle$. The generating function of the second sequence is:

$$\sum_n \binom{r+1}{n-m} z^n = \sum_n \binom{r+1}{n} z^{m+n} = z^m (1+z)^{r+1}\,.$$

For the first one, we recall that, if $\sum_n g_n z^n = G(z)$, then $\sum_n ((-1)^n g_n) z^n = G(-z)$: hence, as the generating function of $\left\langle \binom{r+n}{n} \right\rangle$ is $\dfrac{1}{(1-z)^{r+1}}$, that of $\left\langle (-1)^n \binom{r+n}{n} \right\rangle$ is $\dfrac{1}{(1+z)^{r+1}}$. Then:

$$\sum_n S_n z^n = \frac{1}{(1+z)^{r+1}} \cdot z^m \cdot (1+z)^{r+1} = z^m\,.$$

By comparing coefficients, we get:

$$\sum_{k=0}^{n}(-1)^k\binom{r+k-1}{k}\binom{r}{n-k-m} = [n=m] \ .$$

Exercise 3 (8 points)

Determine for which integer values $n \geqslant 0$ the number $n^{37} - 3n^{25} + 3n^{13} - n$ is divisible by 338.

Solution. As $338 = 2 \cdot 13^2$ as a product of primes, the number $n^{37} - 3n^{25} + 3n^{13} - n$ is divisible by 338 if and only if it is even and divisible by 169. The first part is easy: the number is the sum of four numbers which are either all even or all odd, so it is even. For the other part, we observe that:

$$\begin{aligned} n^{37} - 3n^{25} + 3n^{13} - n &= n \cdot \left(n^{36} - 3n^{24} + 3n^{12} - 1\right) \\ &= n \cdot (n^{12} - 1)^3 \, . \end{aligned}$$

If n is not divisible by 13, then $n^{12} - 1$ is by Fermat's little theorem, so $(n^{12} - 1)^3$ is divisible by 13^3, thus also by 13^2. If n is divisible by 13, then $n^{12} - 1$ is not, so it must be n that is divisible by 169. In conclusion, $n^{37} - 3n^{25} + 3n^{13} - n$ is divisible by 338 if and only if n is either not divisible by 13, or divisible by 169.

Questions (1 point each, 20 points total)

1. How many regions of the plane can be obtained, at most, by drawing 10 straight lines?

   $L_{10} = \dfrac{10 \cdot 11}{2} + 1 = 56.$

2. Describe the method of the summation factor.

   Given a recurrence of the form $a_n T_n = b_n T_{n-1} + c_n$, we determine a sequence $\langle s_n \rangle$ of nonzero numbers such that $s_n b_n = s_{n-1} a_{n-1}$ for every $n \geqslant 1$, put $U_n = s_n a_n T_n$, and solve $U_n = U_{n-1} + s_n c_n$. Then $T_n = \dfrac{1}{s_n a_n} U_n = \dfrac{1}{s_n a_n}\left(U_0 + \sum_{k=1}^{n} s_k c_k\right)$ for every $n \geqslant 1$.

3. Write a function $u(x)$ such that $\Delta u(x) = \binom{x}{m-1}$, where $m \geqslant 1$ is an integer.

   $u(x) = \binom{x}{m}$ is such a function, because $\binom{x+1}{m} = \binom{x}{m} + \binom{x}{m-1}$.

4. Let $\sum_{j,k\in\mathbb{N}} a_{j,k}$ be an infinite sum. What is a sufficient condition to have $\sum_{j\to\infty}\sum_{k\to\infty} a_{j,k} = \sum_{k\to\infty}\sum_{j\to\infty} a_{j,k}$?

   Each of the following answers is valid:

   (a) *That $\sum_{j,k} a_{j,k}$ converges absolutely. (Fubini)*

   (b) *That the $a_{j,k}$ are all nonnegative. (Tonelli)*

5. True or false: for every $x \in \mathbb{R}$, $\lceil e^{\lceil x\rceil}\rceil = \lceil e^x\rceil$.

   False: for $x = 1/2$ it is $\lceil e^{\lceil x\rceil}\rceil = \lceil e\rceil = 3$ but $\lceil e^x\rceil = \lceil\sqrt{e}\rceil = 2$. The function e^x does not satisfy the requirement that, if e^x is integer, then so is x: for example, $\ln 2$ is positive and smaller than 1.

6. What is $\dfrac{15}{4} \bmod \dfrac{3}{2}$?

   $$\frac{15}{4} \bmod \frac{3}{2} = \frac{15}{4} - \frac{3}{2}\cdot\left\lfloor\frac{15/4}{3/2}\right\rfloor = \frac{15}{4} - \frac{3}{2}\cdot\left\lfloor\frac{5}{2}\right\rfloor = \frac{15}{4} - \frac{3}{2}\cdot 2 = \frac{3}{4}.$$

   Alternatively: $\dfrac{15}{4} \bmod \dfrac{3}{2} = \dfrac{15}{4} \bmod \dfrac{6}{4} = \dfrac{3}{4}$.

7. What is a Carmichael number?

   A composite number n satisfying $b^{n-1} \equiv 1 \pmod{n}$ for every $b \geqslant 1$ such that $\gcd(b, n) = 1$.

8. Is $25^{66} - 1$ divisible by 23?

   Yes: as 23 is prime and $25 = 5^2$, $25^{66} - 1 = (25^{22} - 1)\cdot(1 + 25^{22} + 25^{44})$ is divisible by 23 by Fermat's little theorem.

9. Let p and q be distinct primes. What is the value $\mu(pq)$ of Möbius's function μ on the value pq?

   1. In general, if n is the product of k distinct primes, then $\mu(n) = (-1)^k$.

10. True or false: for every $c \in \mathbb{C}$ and $n \geqslant 0$ integer, $\sum_{k=0}^{n}\binom{c}{k}\binom{-c}{n-k} = [n = 0]$.

True: by Vandermonde's identity with $r = c$ and $s = -c$,
$\sum_{k=0}^{n} \binom{c}{k}\binom{-c}{n-k} = \binom{0}{n} = [n = 0]$.

11. Let $G(z)$ be a power series with center 0 and convergence radius 1. Suppose $G(z)$ converges at $z = 1$. What can be said of $\lim_{x \to 1^-} G(x)$?

    The two values are equal by Abel's summation formula.

12. Write the recurrence equation for the Stirling numbers of the first kind.

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

13. How many ways are there to arrange 7 objects into 2 nonempty sets?

$$\begin{Bmatrix} 7 \\ 2 \end{Bmatrix} = 2^{7-1} - 1 = 63\,.$$

14. What is $\lim_{n\to\infty} H_n^{(2)}$?

    $\lim_{n\to\infty} H_n^{(2)} = \sum_{k \geqslant 1} \dfrac{1}{k^2} = \zeta(2) = \dfrac{\pi^2}{6}$.

15. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\langle g_n\,[n \text{ is odd}] \rangle$?

    $\dfrac{G(z) - G(-z)}{2}$.

16. What is the generating function of the sequence of the harmonic numbers?

    $\sum_{n \geqslant 0} H_n z^n = \dfrac{1}{1-z} \ln \dfrac{1}{1-z}$. *In general, if $G(z)$ is the generating function of $\langle g_n \rangle$, then $G(z)/(1-z)$ is the generating function of $\langle \sum_{k=0}^{n} g_k \rangle$*

17. Write the generating function of the sequence $\left\langle \begin{bmatrix} m \\ n \end{bmatrix} \right\rangle$, where $m \geqslant 0$ is an integer.

$$\sum_{n \geqslant 0} \begin{bmatrix} m \\ n \end{bmatrix} z^n = z^{\overline{m}}\,.$$

18. Let $\langle g_n \rangle$ be a sequence which has an analytic generating function. Does $\langle g_n \rangle$ also have an analytic exponential generating function?

Yes: if $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} < \infty$, *then* $\limsup_{n \geqslant 0} \sqrt[n]{|g_n/n!|} < \infty$ *too.*

19. True or false: for any $f(n)$ and $g(n)$, $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$.

True: if $|h(n)| \leqslant C \cdot |f(n)|$ *and* $|k(n)| \leqslant K \cdot |g(n)|$, *then* $|h(n) \cdot k(n)| \leqslant C \cdot K \cdot |f(n) \cdot g(n)|$.

20. True or false: $\sqrt{1 + 1/n} = 1 + \dfrac{1}{2n} - \dfrac{1}{8n^2} + O\left(\dfrac{1}{n^3}\right)$.

True, because $(1+z)^{1/2} = \sum_{n \geqslant 0} \binom{1/2}{n} z^n$, $\binom{1/2}{1} = \frac{1}{2}$, $\binom{1/2}{2} = -\frac{1}{8}$, *and* $\frac{1}{n} \prec 1$.

(Corrects a wrong version where $O(1/n)$ appeared on the left-hand side in place of $1/n$.)