ITT9132 Concrete Mathematics
Final exam

Third date, 12 June 2019

Full name: Code:

1. Take note of the code near your full name: it will be used to display the results.
2. Write your solutions under the corresponding exercise. For Exercises 1, 2 and 3, explaining your reasoning.
3. You may use any formula seen in classroom or appearing in the self-evaluation tests.
4. You may use the additional paper to draft your answers. However, only what is written in the exercises' pages will be evaluated.
5. Partially completed exercises may receive a fraction of the total score.
6. Only handwritten notes are allowed.
7. Electronic devices, including mobile phones must be turned off. Using a pocket or tabletop calculator is allowed as the only exception.
8. It is permitted to leave the room once, for a maximum of 5 minutes, one at a time, handing the assignment to the instructor, who will give it back on return.

Exercise 1 (12 points)

Solve the recurrence:

$$g_n \quad = \quad g_{n-1} - \frac{1}{4}\, g_{n-2}$$

with the initial conditions $g_0 = 1, g_1 = 2$.

Solution. The recurrence is a second-order homogeneous linear recurrence, which is solved easily with generating functions and the Rational Expansion Theorem. Let us follow step by step:

1. We rewrite the recurrence so that it holds for every $n \in \mathbb{Z}$, with the convention that $g_n = 0$ if $n < 0$. This will require to add correction terms c_0 and c_1 to keep track of the initial conditions. The recurrence becomes:

   $$g_n = g_{n-1} - \frac{1}{4} g_{n-2} + c_0\,[n = 0] + c_1\,[n = 1] \quad \text{for every } n \in \mathbb{Z}$$

   Now:

   - For $n = 0$ we have $g_0 = 1$ but $g_{-1} - \frac{1}{4} g_{-2} = 0$. Thus, $c_0 = 1$.
   - For $n = 0$ we have $g_1 = 2$ but $g_0 - \frac{1}{4} g_{-1} = 1$. Thus, $c_1 = 1$.

2. By multiplying by z^n and summing over $n \in \mathbb{Z}$ we obtain:

   $$\sum_n g_n z^n = \sum_n g_{n-1} z^n - \frac{1}{4} \sum_n g_{n-2} z^n + \sum_n [n = 0]\, z^n + \sum_n [n = 1]\, z^n$$

   which, calling $G(z)$ the left-hand side, becomes:

   $$G(z) = zG(z) - \frac{1}{4} z^2 G(z) + 1 + z\,.$$

3. By solving with respect to $G(z)$ we obtain:

   $$G(z) = \frac{1 + z}{1 - z + z^2/4}\,.$$

4. Let $P(z)$ and $Q(z)$ be the numerator and denominator in the expression of $G(z)$. The reflected polynomial of the denominator is $Q^R(z) = z^2 - z + 1/4 = (z - 1/2)^2$: then $Q(z) = (1 - z/2)^2$. In the notation of the Rational Expansion Theorem we have $\ell = 1$, $\rho_1 = 1/2$ and $d_1 = 2$, so the solution will have the form $g_n = (a_1 n + b_1) \cdot (1/2)^n$. To compute a_1 we can use either of the following formulas:
$$a_1 = \frac{P(1/\rho_1)}{(d_1 - 1)! \prod_{j \neq 1} (1 - \rho_j/\rho_1)^{d_j}} = \frac{1 + 1/(1/2)}{1! \cdot \text{(empty product)}} = 3\,;$$
$$a_1 = \frac{(-\rho_1)^{d_1} P(1/\rho_1) d_1}{Q^{(d_1)}(1/\rho_1)} = \frac{(-1/2)^2 (1 + 1/(1/2)) \cdot 2}{1/2} = 3\,.$$
To determine b_1, we put $n = 0$ and we obtain $b_1 = g_0 = 1$.

In conclusion,
$$g_n = \frac{3n+1}{2^n}\,.$$

Exercise 2 (10 points)

Let $r, s \in \mathbb{R}$. For $n \geqslant 0$ integer compute:
$$S_n = \sum_{i+j+k=n} (-1)^i \binom{i+r+s-1}{i} \binom{r}{j} \binom{s}{k}\,.$$

Solution. The sequence $\langle S_n \rangle$ is the convolution of the three sequences $\left\langle (-1)^n \binom{n+r+s-1}{n} \right\rangle$, $\left\langle \binom{r}{n} \right\rangle$, and $\left\langle \binom{s}{n} \right\rangle$. The generating functions of the last two sequences are $(1+z)^r$ and $(1+z)^s$, respectively. For the first one, we recall that multiplying the nth term by $(-1)^n$ corresponds to evaluating the generating function in $-z$ instead of z: as the generating function of $\left\langle \binom{n+r+s-1}{n} \right\rangle$ is $\frac{1}{(1-z)^{r+s}}$, that of $\left\langle (-1)^n \binom{n+r+s-1}{n} \right\rangle$ is $\frac{1}{(1+z)^{r+s}}$. Then:
$$\begin{aligned} \sum_{n \geqslant 0} S_n z^n &= \frac{1}{(1+z)^{r+s}} \cdot (1+z)^r \cdot (1+z)^s \\ &= 1\,. \end{aligned}$$

By comparing the coefficients, we find:
$$\sum_{i+j+k=n} (-1)^i \binom{i+r+s-1}{i} \binom{r}{j} \binom{s}{k} = [n = 0]\,.$$

Exercise 3 (8 points)

Determine for which integer values $n \geqslant 0$ the number

$$M = n^{11} - n^7 - n^5 + n$$

is divisible by 70.

Solution. As $70 = 2 \cdot 5 \cdot 7$, M is divisible by 70 if and only if it is divisible by 2, 5 and 7. Now, M is the sum of four powers of the same number, which are either all even or all odd: in either case, M is even. Moreover,

$$\begin{aligned} n^{11} - n^7 - n^5 + n &= n(n^{10} - n^6 - n^4 + 1) \\ &= n(n^6(n^4 - 1) - (n^4 - 1)) \\ &= n(n^6 - 1)(n^4 - 1)\,. \end{aligned}$$

If n is not divisible by 5, then $n^4 - 1$ is by Fermat's little theorem; similarly, if n is not divisible by 7, then $n^6 - 1$ is. This means that, whatever n is, M has at least one factor divisible by 5 and at least one factor divisible by 7. In conclusion, M is divisible by 70 for every $n \geqslant 0$.

Questions (1 point each, 20 points total)

1. Let $f(1) = 1, f(2n) = 2f(n) + 1, f(2n + 1) = 2f(n)$ for every $n \geqslant 1$. Compute $f(24)$.

   This is a "Josephus-like" problem in base 2 with $\alpha = 1, \beta_0 = 1, \beta_1 = 0$; hence, $f(24) = f((11000)_2) = (1\beta_1\beta_0\beta_0\beta_0)_2 = (10111)_2 = 23$.

2. Write the formula of summation by parts.

   $u\Delta v = \Delta(uv) - Ev\Delta u$ *where* $Ev(x) = v(x + 1)$.

3. What function $u(x)$ satisfies $\Delta u(x) = \dfrac{1}{x+1}$ and $u(0) = 0$?

   $u(x) = H_x$.

4. Write an infinite sum which does not have the commutative property.

   $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$ *but* $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2} \ln 2$.

5. True or false: for every $x > 0$, $\left\lfloor -\sqrt{\lceil x \rceil} \right\rfloor = \lfloor -\sqrt{x} \rfloor$.

   True: $\left\lfloor -\sqrt{\lceil x \rceil} \right\rfloor = -\left\lceil \sqrt{\lceil x \rceil} \right\rceil = -\lceil \sqrt{x} \rceil = \lfloor -\sqrt{x} \rfloor$.

6. State Bézout's theorem.

   The greatest common divisor $\gcd(a, b)$ *of two integers* a, b *is the smallest positive linear combination of* a *and* b *with integer coefficients.*

7. Give the definition of multiplicative function.

   A function f *defined on the positive integers is multiplicative if* $f(mn) = f(m)f(n)$ *for every* m, n *such that* $\gcd(m, n) = 1$.

8. State Euler's theorem.

   If a *and* m *are positive integers and* $\gcd(a, m) = 1$, *then* $a^{\phi(m)} \equiv 1 \pmod{m}$, *where* ϕ *is Euler's totient function.*

9. Let μ be the Möbius function. What is $\mu(2856783433768967893700)$? *Hint:* This is a "don't panic" question.

   0. As 2856783433768967893700 ends with two zeros, it is divisible by 100: but if n *is divisible by a perfect square, then* $\mu(n) = 0$.

10. True or false: for every $r \in \mathbb{R}$ and $k, m \in \mathbb{Z}$, $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$.

    True: the equality holds for every $r \geqslant m$ *integer, thus also for every* r *real by the polynomial argument.*

11. Write a power series $G(z) = \sum_{n \geqslant 0} g_n z^n$ such that $\lim_{x \to 1^-} G(x)$ exists, but $G(1)$ does not.

    $G(z) = \sum_{n \geqslant 0} (-1)^n z^n$ *is such a power series, because* $\lim_{x \to 1^-} \sum_{n \geqslant 0} (-1)^n x^n = \lim_{x \to 1^-} \dfrac{1}{1 + x} = \dfrac{1}{2}$, *but* $\sum_{n \geqslant 0} (-1)^n$ *does not exist.*

12. True or false: for every $n \geqslant 0$, $\sum_{k=0}^{n} \left[{n \atop k} \right] = n!$.

    True. The partitions of $[1 : n]$ *into nonempty cycles are in bijection with the permutations of* n *objects.*

13. Write the generalized Cassini's identity.

    $f_{n+k} = f_{n+1} f_k + f_n f_{k-1}$ *for every* $n, k \in \mathbb{Z}$.

14. Let $\langle B_n \rangle$ be the sequence of Bernoulli numbers. What is $\limsup_{n\geqslant 0} \sqrt[n]{|B_n|}$?

$\limsup_{n\geqslant 0} \sqrt[n]{|B_n|} = +\infty$, *because* $|B_{2n}| \approx \left(\dfrac{n}{\pi e}\right)^{2n}$.

15. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\langle n g_n \rangle$?

$zG'(z)$.

16. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\left\langle \sum_{i+j+k+\ell=n} g_i g_j g_k g_\ell \right\rangle$?

$(G(z))^4$. *The sequence is the convolution of four copies of* $\langle g_n \rangle$.

17. What is the number of complete binary trees with 6 leaves?

$C_5 = \frac{1}{6}\binom{10}{5} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5} = 42.$

18. Write the exponential generating function of the sequence of natural numbers.

ze^z. *If* $\widehat{G}(z)$ *is the exponential generating function of* $\langle g_n \rangle$, *then that of* $\langle n g_n \rangle$ *is* $z\widehat{G}(z)$.

19. True or false: for any $f(n)$ and $g(n)$, $O(f(n)+g(n)) = f(n)+O(g(n))$.

False: if $f(n) = n$ *and* $g(n) = 1$, *then* $h(n) = 2n$ *belongs to the first class, but not to the second.*

20. True or false: $e^{1/\sqrt{n}} = 1 + \dfrac{1}{\sqrt{n}} + \dfrac{1}{2n} + \dfrac{1}{6n^{3/2}} + O\left(\dfrac{1}{n^2}\right)$.

True, because $e^z = \sum_{n\geqslant 0} \dfrac{z^n}{n!}$, $\dfrac{1}{\sqrt{n}} \prec 1$, *and* $\left(\dfrac{1}{\sqrt{n}}\right)^4 = \dfrac{1}{n^2}$.