Recurrent Problems

ITT9132 Concrete Mathematics
Lecture 3 – 11 February 2019

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Next section

Binary expansion of $n = 2^m + \ell$

Denote

$$n = (b_m b_{m-1} \dots b_1 b_0)_2$$

where $b_i \in \{0, 1\}$ and $b_m = 1$.

This notation stands for

$$n = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

For example

$$20 = (10100)_2 \quad \text{and} \quad 83 = (1010011)_2$$

TAL TECH

Binary expansion of $n = 2^m + \ell$, where $0 \leqslant \ell < 2^m$

Observations:

1. $\ell = (0 b_{m-1} \dots b_1 b_0)_2$.
2. $2\ell = (b_{m-1} \dots b_1 b_0 0)_2$.
3. $2^m = (10 \dots 00)_2$ and $1 = (00 \dots 01)_2$.
4. $n = 2^m + \ell = (1 b_{m-1} \dots b_1 b_0)_2$.
5. $2\ell + 1 = (b_{m-1} \dots b_1 b_0 1)_2$

Corollary

$$J((\boxed{1}\, b_{m-1} \dots b_1 b_0)_2 = (b_{m-1} \dots b_1 b_0\, \boxed{1})_2$$

shift

Binary expansion of $n = 2^m + \ell$, where $0 \leqslant \ell < 2^m$

Observations:

1. $\ell = (0 b_{m-1} \ldots b_1 b_0)_2$.
2. $2\ell = (b_{m-1} \ldots b_1 b_0 0)_2$.
3. $2^m = (10 \ldots 00)_2$ and $1 = (00 \ldots 01)_2$.
4. $n = 2^m + \ell = (1 b_{m-1} \ldots b_1 b_0)_2$.
5. $2\ell + 1 = (b_{m-1} \ldots b_1 b_0 1)_2$

Corollary

$$J((\boxed{1}\ b_{m-1} \ldots b_1 b_0)_2 = (b_{m-1} \ldots b_1 b_0\ \boxed{1})_2$$

shift

Binary expansion of $n = 2^m + \ell$, where $0 \leqslant \ell < 2^m$

Example

$$\begin{aligned} 100 &= 64 + 32 + 4 \\ J(100) &= J((1100100)_2) = (1001001)_2 \\ J(100) &= 64 + 8 + 1 = 73 \end{aligned}$$

Iterating the Josephus function

Consider a sequence $x_0, x_1, \ldots, x_k, \ldots$ where:

- $x_0 = n$ is an arbitrary positive integer; and
- $x_k = J(x_{k-1})$ for every $k \geq 1$.

Questions:

1. Will the sequence reach a fixed point?
   That is: will $x_{k+1} = x_k$ for every k large enough?
2. If so: what are the possible fixed points?

Iterating the Josephus function: the answer

Proposition A

For every positive integer n, the sequence defined by:

$$\begin{aligned} x_0 &= n, \\ x_k &= J(x_{k-1}) \ \forall k \geqslant 1 \end{aligned}$$

reaches the fixed point $2^{\nu(n)} - 1$, where $\nu(n)$ is the number of bits equal to 1 in the binary representation of n.

Iterating the Josephus function: the answer

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Proof that x_n reaches a fixed point:

- For every $n = 2^m + \ell$ we have $J(n) = 2\ell + 1 \leqslant n$.
- Then the sequence x_k is *nonincreasing* in k:
  If $k \leqslant m$, then $x_k \geqslant x_m$.
- But a nonincreasing sequence of positive integers is ultimately constant.

Iterating the Josephus function: the answer

Proposition A

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reaches the fixed point $2^{\nu(n)} - 1$, where $\nu(n)$ is the number of bits equal to 1 in the binary representation of n.

Proof that the fixed point is $2^{\nu(n)} - 1$:

- The binary representation of $J(n)$ is obtained from that of n by a circular permutation.
- But after such a permutation, a leading 0 disappears, while a leading 1 is preserved.
- Then the binary writing of any fixed point must be made entirely of 1s.

Next section

Generalization

Josephus function $J : \mathbb{N} \longrightarrow \mathbb{N}$

was defined using recurrences:

$$\begin{aligned} J(1) &= 1\,; \\ J(2n) &= 2J(n) - 1 \ \text{ for } n \geqslant 1\,; \\ J(2n+1) &= 2J(n) + 1 \ \text{ for } n \geqslant 1\,. \end{aligned}$$

Introducing integer constants α, β and γ, generalize it as follows:

$$\begin{aligned} J(1) &= \alpha\,; \\ J(2n) &= 2J(n) + \beta \ \text{ for } n \geqslant 1\,; \\ J(2n+1) &= 2J(n) + \gamma \ \text{ for } n \geqslant 1\,. \end{aligned}$$

Our $J(n)$ corresponds to $\alpha = 1$, $\beta = -1$, $\gamma = 1$.

The repertoire method

To find closed form of a function f:

Step 1 Find few initial values for f.

Step 2 Find (or guess) closed formula from the values found by Step 1:
examine a repertoire of cases and combine them to find general closed formula.

Step 3 Verify the closed formula constructed as the result of Step 2.

The idea is to examine a repertoire of cases and use it to find a general closed formula for the recurrently defined function.

Repertoire method for generalized f: STEP 1

n	$f(n)$	Calculation
1	α	$f(1) = \alpha$
2	$2\alpha + \beta$	$f(2) = 2f(1) + \beta$
3	$2\alpha + \gamma$	$f(3) = 2f(1) + \gamma$
4	$4\alpha + 3\beta$	$f(4) = 2f(2) + \beta$
5	$4\alpha + 2\beta + \gamma$	$f(5) = 2f(2) + \gamma$
6	$4\alpha + \beta + 2\gamma$	$f(6) = 2f(3) + \beta$
7	$4\alpha + 3\gamma$	$f(7) = 2f(3) + \gamma$
8	$8\alpha + 7\beta$	$f(8) = 2f(4) + \beta$
9	$8\alpha + 6\beta + \gamma$	$f(9) = 2f(4) + \gamma$

Repertoire method for generalized f: STEP 2

Observations:

For $n = 1, 2, \ldots, 9$, taking $n = 2^k + \ell$:

- The coefficient of α is 2^k;
- The coefficient of β is $2^k - 1 - \ell$;
- The coefficient of γ is ℓ.

Repertoire method for generalized f: STEP 3

Proposition

If the function f is defined by the recurrence formula:

$$\begin{aligned} f(1) &= \alpha ; \\ f(2n) &= 2J(n)+\beta \text{ for } n \geqslant 1 ; \\ f(2n+1) &= 2J(n)+\gamma \text{ for } n \geqslant 1 . \end{aligned}$$

then letting $n = 2^k + \ell$,

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma,$$

where:

$$\begin{aligned} A(n) &= 2^k ; \\ B(n) &= 2^k - 1 - \ell ; \\ C(n) &= \ell . \end{aligned}$$

Proof of the Proposition (1)

Lemma 1. $A(n) = 2^k$, where $n = 2^k + \ell$ and $0 \leqslant \ell < 2^k$.

Proof.

Let $\alpha = 1$ and $\beta = \gamma = 0$. Then $f(n) = A(n)$ and:

$$A(1) = 1\,; \quad A(2n) = 2A(n) \text{ for } n > 0\,; A(2n+1) = 2A(n) \text{ for } n > 0\,.$$

Proof by induction over k:

Basis: If $k = 0$, then $n = 2^0 + \ell$ and $0 \leqslant \ell < 1$. Thus $n = 1$ and

$$A(1) = 2^0 = 1\,.$$

Step: Let us assume that $A(2^{k-1} + t) = 2^{k-1}$, where $0 \leqslant t < 2^{k-1}$ Two cases:

- If n is even, then ℓ is even and $\ell/2 < 2^{k-1}$, thus

$$A(n) = A(2^k + \ell) = 2A(2^{k-1} + \ell/2) = 2 \cdot 2^{k-1} = 2^k$$

- If n is odd, then $\ell - 1$ is even and $(\ell - 1)/2 < 2^{k-1}$, thus

$$A(n) = A(2^k + \ell) = 2A(2^{k-1} + (\ell - 1)/2) = 2 \cdot 2^{k-1} = 2^k$$

Proof of the Proposition (2)

Lemma 2. $A(n) - B(n) - C(n) = 1$, for all $n \in \mathbb{N}$.

Proof.

Let f be the constant function $f(n) = 1$. Then:

$$f(1) = \alpha; \quad f(2n) = 2f(n) + \beta; \quad f(2n+1) = 2f(n) + \gamma$$

or equivalently,

$$1 = \alpha; \quad 1 = 2 + \beta; \quad 1 = 2 + \gamma.$$

As this must hold for every $n \geq 1$, it must be $\alpha = 1$ and $\beta = \gamma = -1$. □

Proof of the Proposition (3)

Lemma 3. $A(n) + C(n) = n$, for all $n \in \mathbb{N}$.

Proof.

Let $f(n) = n$. Then:

$$f(1) = \alpha; \quad f(2n) = 2f(n) + \beta; \quad f(2n+1) = 2f(n) + \gamma$$

or equivalently,

$$1 = \alpha; \quad 2n = 2n + \beta; \quad 2n+1 = 2n + \gamma.$$

As this must hold for every $n \geq 1$, it must be $\alpha = 1$, $\beta = 0$ and $\gamma = 1$. □

Proof of the Proposition (4)

From Lemma 3 and Lemma 1 we can conclude:

$$2^k + C(n) = A(n) + C(n) = n = 2^k + \ell\,,$$

which gives:

$$C(n) = \ell\,.$$

From Lemma 2 follows:

$$B(n) = A(n) - 1 - C(n) = 2^k - 1 - \ell\,.$$

□

Next section

The repertoire method: Basic ideas

Let the recursion scheme

$$\begin{aligned} g(0) &= \alpha\,, \\ g(n+1) &= \Phi(g(n)) + \Psi(n;\beta,\gamma,\ldots) \quad \text{for } n \geq 0\,. \end{aligned}$$

have the following properties:

1. Φ is linear in g: if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$, then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
   No hypotheses are made on the dependence of g on n.
2. Ψ is linear in each of the $m-1$ parameters $\beta,\gamma,\ldots$
   No hypotheses are made on the dependence of Ψ on n.

Then the whole system is linear in the parameters $\alpha,\beta,\gamma,\ldots$
We can then look for a general solution of the form

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \ldots$$

The repertoire method: Description

Suppose we have a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfying the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$
2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then the functions $A(n), B(n), C(n), \ldots$ are uniquely determined.
The reason is that, for every fixed n,

$$\begin{array}{lllll} \alpha_1 A(n) & +\beta_1 B(n) & +\gamma_1 C(n) & +\ldots & = & g_1(n) \\ \vdots & & & & = & \vdots \\ \alpha_m A(n) & +\beta_m B(n) & +\gamma_m C(n) & +\ldots & = & g_m(n) \end{array}$$

is a system of m linear equations in the m unknowns $A(n), B(n), C(n), \ldots$ whose coefficients matrix is invertible.

Next section

Binary representation of generalized Josephus function

Definition

The generalized Josephus function (GJ-function) is defined for α, β_0, β_1 as follows:

$$\begin{aligned} f(1) &= \alpha \\ f(2n+j) &= 2f(n)+\beta_j \text{ for } j=0,1 \text{ and } n>0\,. \end{aligned}$$

We obtain the definition used before if to select $\beta_0 = \beta$ and $\beta_1 = \gamma$.

Binary representation of generalized Josephus function (2)

Case A: Argument is even

If $2n = 2^m + \ell$, then the binary notation is

$$2n = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 0$ and $b_m = 1$.

Hence

$$n = b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

Binary representation of generalized Josephus function (3)

Case B: Argument is odd

If $2n+1 = 2^m + \ell$, then the binary notation is

$$2n+1 = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n+1 = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 1$ and $b_m = 1$.

We get

$$\begin{aligned} 2n+1 &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + 1 \\ 2n &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 \\ n &= b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1 \end{aligned}$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

Binary representation of generalized Josephus function (3)

Case B: Argument is odd

If $2n+1 = 2^m + \ell$, then the binary notation is

$$2n+1 = (b_m b_{m-1} \ldots b_1 b_0)_2$$

or

$$2n+1 = b_m 2^m + b_{m-1} 2^{m-1} + \ldots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 1$ and $b_m = 1$.

We get

$$\begin{aligned} 2n+1 &= b_m 2^m + b_{m-1} 2^{m-1} + \ldots + b_1 2 + 1 \\ 2n &= b_m 2^m + b_{m-1} 2^{m-1} + \ldots + b_1 2 \\ n &= b_m 2^{m-1} + b_{m-1} 2^{m-2} + \ldots + b_2 2 + b_1 \end{aligned}$$

Same results for cases A and B indicates that we don't need to consider even and odd cases separately.

Binary representation of generalized Josephus function (4)

Let's evaluate:

$$
\begin{aligned}
f((b_m, b_{m-1}, \ldots, b_1, b_0)_2) &= 2f((b_m, b_{m-1}, \ldots, b_1)_2) + \beta_{b_0} \\
&= 2 \cdot (2f((b_m, b_{m-1}, \ldots, b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, \ldots, b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= f((b_m)_2)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= f(1)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \, ,
\end{aligned}
$$

where

$$
\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}
$$

$$
f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2
$$

Binary representation of generalized Josephus function (4)

Let's evaluate:

$$
\begin{aligned}
f((b_m, b_{m-1}, \ldots, b_1, b_0)_2) &= 2f((b_m, b_{m-1}, \ldots, b_1)_2) + \beta_{b_0} \\
&= 2 \cdot (2f((b_m, b_{m-1}, \ldots, b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, \ldots, b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= f((b_m)_2)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= f(1)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \,,
\end{aligned}
$$

where

$$
\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}
$$

$$
f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2
$$

Example

Original Josephus function: $\alpha = 1$, $\beta_0 = -1$, $\beta_1 = 1$ i.e.

$$\begin{aligned} f(1) &= 1 \\ f(2n) &= 2f(n) - 1 \\ f(2n+1) &= 2f(n) + 1 \end{aligned}$$

Compute

$$f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2$$

$$\begin{aligned} f(100) = f((1100100)_2) &= (1, 1, -1, -1, 1, -1, -1)_2 \\ &= 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73 \end{aligned}$$

Generalized Josephus function: Multiple bases

Let $c, d \geqslant 2$ be integers.
Consider the following recurrent problem:

$$\begin{array}{rcll} f(j) & = & \alpha_j & \text{for } 1 \leqslant j < d\,; \\ f(dn+j) & = & cf(n) + \beta_j & \text{for } 0 \leqslant j < d \text{ and } n \geqslant 1\,. \end{array}$$

How can we compute $f(n)$ for an arbitrary positive integer n, without having to go through the entire iterative process?

Multiple bases representation

We can actually use the same technique!

Let $(b_m b_{m-1} \dots b_1 b_0)_d$ be the base-d writing of n. Then $b_m \neq 0$ and:

$$\begin{aligned}
f((b_m, b_{m-1}, \dots, b_1, b_0)_d) &= cf((b_m, b_{m-1}, \dots, b_1)_d) + \beta_{b_0} \\
&= c \cdot (cf((b_m, b_{m-1}, \dots, b_2)_d) + \beta_{b_1}) + \beta_{b_0} \\
&= c^2 f((b_m, b_{m-1}, \dots, b_2)_d) + c\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= c^m \cdot f(b_m) + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0} \\
&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}$$

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Let $(b_m b_{m-1} \dots b_1 b_0)_d$ be the base-d writing of n. Then $b_m \neq 0$ and:

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\begin{aligned}
f((b_m, b_{m-1}, \dots, b_1, b_0)_d) &= cf((b_m, b_{m-1}, \dots, b_1)_d) + \beta_{b_0} \\
&= c \cdot (cf((b_m, b_{m-1}, \dots, b_2)_d) + \beta_{b_1}) + \beta_{b_0} \\
&= c^2 f((b_m, b_{m-1}, \dots, b_2)_d) + c\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= c^m \cdot f(b_m) + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0} \\
&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}
$$

With a slight abuse of notation: (the β_i's need not be base-c digits)

$$f((b_m b_{m-1} \dots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \dots \beta_{b_1} \beta_{b_0})_c$$

Multiple bases representation

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Let $(b_m b_{m-1} \ldots b_1 b_0)_d$ be the base-d writing of n. Then $b_m \neq 0$ and:

$$\begin{aligned} f((b_m, b_{m-1}, \ldots, b_1, b_0)_d) &= cf((b_m, b_{m-1}, \ldots, b_1)_d) + \beta_{b_0} \\ &= c \cdot (cf((b_m, b_{m-1}, \ldots, b_2)_d) + \beta_{b_1}) + \beta_{b_0} \\ &= c^2 f((b_m, b_{m-1}, \ldots, b_2)_d) + c\beta_{b_1} + \beta_{b_0} \\ &= \vdots \\ &= c^m \cdot f(b_m) + c^{m-1}\beta_{b_{m-1}} + \ldots + c\beta_{b_1} + \beta_{b_0} \\ &= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \ldots + c\beta_{b_1} + \beta_{b_0} \end{aligned}$$

Or, more precisely:

$$f((b_m b_{m-1} \ldots b_1 b_0)_d) = p(c) \text{ where } p(x) = \alpha_{b_m} x^m + \beta_{b_{m-1}} x^{m-1} + \ldots + \beta_{b_1} x + \beta_{b_0}$$

Next section

Sequences

Definition

A sequence of elements of a set A is a function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is the set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$;
- $\{a_n\}_{n \in \mathbb{N}}$;
- $\langle a_0, a_1, a_2, a_3, \ldots \rangle$.

a_n is called the *n*th term of the sequence f

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Example

$$a_0 = 0,\ a_1 = \frac{1}{2\cdot 3},\ a_2 = \frac{2}{3\cdot 4},\ a_3 = \frac{3}{4\cdot 5},\cdots$$

or

$$\left\langle 0, \frac{1}{6}, \frac{1}{6}, \frac{3}{20}, \frac{2}{15}, \cdots, \frac{n}{(n+1)(n+2)}, \cdots \right\rangle$$

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- $\{a_n\}_{n\in\mathbb{N}}$;
- $\langle a_0, a_1, a_2, a_3, \ldots\rangle$.

a_n is called the nth term of the sequence f

Notation

$$f(n) = \frac{n}{(n+1)(n+2)}$$

or

$$a_n = \frac{n}{(n+1)(n+2)}$$

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Sets of indices

- $\mathbb{N}$ – set of indices of the sequence $f = \{a_n\}_{n\in\mathbb{N}}$
- Any countably infinite set can be used as an index set. Examples of other frequently used indices are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$.
  - $\mathbb{N} - K$, where K is any finite subset of $\mathbb{N}$.
  - The set $\mathbb{Z}$ of relative integers.
  - $\{1, 3, 5, 7, \ldots\} = \mathrm{Odd}$.
  - $\{0, 2, 4, 6, \ldots\} = \mathrm{Even}$.

TAL TECH

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  - $\{1,3,5,7,\dots\} = \mathrm{Odd}$.
  - $\{0,2,4,6,\dots\} = \mathrm{Even}$.

$A \sim B$ denotes that sets A and B are of the same cardinality, i.e. $|A| = |B|$.

Sets of indices

- $\mathbb{N}$ – set of indices of the sequence $f = \{a_n\}_{n \in \mathbb{N}}$
- Any countably infinite set can be used as an index set. Examples of other frequently used indices are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$.
  - $\mathbb{N} - K$, where K is any finite subset of $\mathbb{N}$.
  - The set $\mathbb{Z}$ of relative integers.
  - $\{1, 3, 5, 7, \ldots\} = \mathrm{Odd}$.
  - $\{0, 2, 4, 6, \ldots\} = \mathrm{Even}$.

Two sets A and B have the same cardinality if there exists a bijection, that is, an injective and surjective function, from A to B.

(See http://www.mathsisfun.com/sets/injective-surjective-bijective.html for detailed explanation)

Finite sequences

- A finite sequence of elements of a set A is a function
  $f : K \to A$,
  where K is set a finite subset of natural numbers

  For example: $f : \{1, 2, 3, 4, \cdots, n\} \to A$, $n \in \mathbb{N}$

  Special case: $n = 0$, i.e. empty sequence: $f(\emptyset) = e$

Domain of the sequence

$$f : T \to A$$

$$a_n = \frac{n}{(n-2)(n-5)}$$

The domain of f is $T = \mathbb{N} - \{2,5\}$.

Next section

Notation

For a finite set $K = \{1, 2, \cdots, m\}$ and a given sequence $f : K \to \mathbb{R}$ with $f(n) = a_n$ we write

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m$$

Alternative notations

$$\sum_{k=1}^{m} a_k = \sum_{1 \leqslant k \leqslant m} a_k = \sum_{k \in \{1, \cdots, m\}} a_k = \sum_{K} a_k$$

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Warmup: What does this notation mean?

$$\sum_{k=4}^{0} q_k$$

Options:

1. $\sum_{k=4}^{0} q_k = q_4 + q_3 + q_2 + q_1 + q_0 = \sum_{k \in \{4,3,2,1,0\}} q_k = \sum_{k=0}^{4} q_k$.
   This seems the sensible thing—**but:**
2. $\sum_{4 \leq k \leq 0} q_k = 0$ also looks like a feasible interpretation—**but:**
3. If

$$\sum_{k=m}^{n} q_k = \sum_{k \leq n} q_k - \sum_{k < m} q_k ,$$

(provided the two sums on the right-hand side exist finite) then $\sum_{k=4}^{0} q_k = \sum_{k \leq 0} q_k - \sum_{k<4} q_k = -q_1 - q_2 - q_3$.

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Warmup: Interpreting the Σ-notation

Compute $\sum_{\{0\le k\le 5\}} a_k$ and $\sum_{\{0\le k^2\le 5\}} a_{k^2}$.

First sum

$$\{0 \le k \le 5\} = \{0,1,2,3,4,5\} :$$

thus, $\sum_{\{0\le k\le 5\}} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

Second sum

$$\{0 \le k^2 \le 5\} = \{0,1,2,-1,-2\} :$$

thus,

$\sum_{\{0\le k\le 5\}} a_{k^2} = a_{0^2} + a_{1^2} + a_{2^2} + a_{(-1)^2} + a_{(-2)^2} = a_0 + 2a_1 + 2a_2$.

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Next section

Sums and Recurrences

Computation of any sum

$$S_n = \sum_{k=1}^{n} a_k$$

can be presented in recursive form:

$$\begin{aligned} S_0 &= a_0 \\ S_n &= S_{n-1} + a_n \end{aligned}$$

$\Rightarrow$ from CHAPTER ONE can be used for finding closed formulas for evaluating sums.

Recalling the repertoire method

- Given

$$\begin{aligned} g(0) &= \alpha \\ g(n) &= \Phi(g(n-1)) + \Psi(\beta,\gamma,\ldots) \text{ for } n > 0\,. \end{aligned}$$

  where Φ and Ψ are linear, for example, if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
- Closed form is :

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \cdots \qquad (1)$$

- Functions $A(n), B(n), C(n), \ldots$ can be found from the system of equations

$$\begin{aligned} \alpha_1 A(n) + \beta_1 B(n) + \gamma_1 C(n) + \cdots &= g_1(n) \\ &= \vdots \\ \alpha_m A(n) + \beta_m B(n) + \gamma_m C(n) + \cdots &= g_m(n) \end{aligned}$$

  where $\alpha_i, \beta_i, \gamma_i \cdots$ are constants committing (1) and recurrence relationship for the repertoire case $g_i(n)$ and any n.

Next subsection

Example 1: arithmetic sequence

Arithmetic sequence: $a_n = a + bn$

Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$:

$$
\begin{aligned}
S_0 &= a \\
S_n &= S_{n-1} + (a + bn)\,, \text{ for } n > 0\,.
\end{aligned}
$$

Let's find a closed form for a bit more general recurrent equation:

$$
\begin{aligned}
R_0 &= \alpha \\
R_n &= R_{n-1} + (\beta + \gamma n)\,, \text{ for } n > 0\,.
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Evaluation of terms $R_n = R_{n-1} + (\beta + \gamma n)$

$$\begin{aligned} R_0 &= \alpha \\ R_1 &= \alpha + \beta + \gamma \\ R_2 &= \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma \\ R_3 &= \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma \end{aligned}$$

Observation

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$A(n), B(n), C(n)$ can be evaluated using repertoire method:
we will consider three cases

1. $R_n = 1$ for all n
2. $R_n = n$ for all n
3. $R_n = n^2$ for all n

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2. $R_n = n$ for all n
3. $R_n = n^2$ for all n

Repertoire method: case 1

Lemma 1: $A(n) = 1$ for all n

- $1 = R_0 = \alpha$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $1 = 1 + (\beta + \gamma n)$. This is possible only when $\beta = \gamma = 0$

Hence

$$1 = A(n) \cdot 1 + B(n) \cdot 0 + C(n) \cdot 0$$

□

Repertoire method: case 2

Lemma 2: $B(n) = n$ for all n

- $\alpha = R_0 = 0$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $n = (n-1) + (\beta + \gamma n)$.
  I.e. $1 = \beta + \gamma n$.
  This gives $\beta = 1$ and $\gamma = 0$

Hence

$$n = A(n) \cdot 0 + B(n) \cdot 1 + C(n) \cdot 0$$

□

Repertoire method: case 3

Lemma 3: $C(n) = \frac{n^2+n}{2}$ for all n

- $\alpha = R_0 = 0^2 = 0$.
- Equation $R_n = R_{n-1} + (\beta + \gamma n)$ can be rewritten as:
  - $n^2 = (n-1)^2 + \beta + \gamma n$.
  - $n^2 = n^2 - 2n + 1 + \beta + \gamma n$.
  - $0 = (1 + \beta) + n(\gamma - 2)$.

  This is valid iff $1 + \beta = 0$ and $\gamma - 2 = 0$

Hence

$$n^2 = A(n) \cdot 0 + B(n) \cdot (-1) + C(n) \cdot 2$$

Due to Lemma 2 we get

$$n^2 = -n + 2C(n)$$

Repertoire method: summing up

According to Lemma 1, 2, 3, we get:

1. $R_n = 1$ for all n $\Longrightarrow$ $A(n) = 1$
2. $R_n = n$ for all n $\Longrightarrow$ $B(n) = n$
3. $R_n = n^2$ for all n $\Longrightarrow$ $C(n) = (n^2 + n)/2$

Hence,

$$R_n = \alpha + n\beta + \left(\frac{n^2 + n}{2}\right)\gamma$$

The sum for arithmetic sequence we obtain taking $\alpha = \beta = a$ and $\gamma = b$:

$$S_n = \sum_{k=0}^{n} (a + bk) = (n+1)a + \frac{n(n+1)}{2} b$$

Next subsection

Perturbation method

Finding the closed form for $S_n = \sum_{0 \leqslant k \leqslant n} a_k$:

- Rewrite S_{n+1} by splitting off first and last term:

$$
\begin{aligned}
S_n + a_{n+1} &= a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k \\
&= a_0 + \sum_{1 \leqslant k+1 \leqslant n+1} a_{k+1} \\
&= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}
\end{aligned}
$$

- Work on last sum and express in terms of S_n.
- Finally, solve for S_n.

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$
\begin{aligned}
S_0 &= a \\
S_n &= S_{n-1} + ax^n \,, \text{ for } n > 0 \,.
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- Splitting off the first term gives

$$\begin{aligned} S_n + a_{n+1} &= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} \\ &= a + \sum_{0 \leqslant k \leqslant n} ax^{k+1} \\ &= a + x \sum_{0 \leqslant k \leqslant n} ax^k \\ &= a + xS_n \end{aligned}$$

- Hence, we have the equation

$$S_n + ax^{n+1} = a + xS_n$$

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- Solution:

$$S_n = \frac{a - ax^{n+1}}{1 - x}$$

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- Solution:

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Closed formula for geometric sum:

$$S_n = \frac{a(x^{n+1} - 1)}{x - 1}$$

Next subsection

Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

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\begin{aligned}
T_0 &= 0 \\
T_n &= 2T_{n-1} + 1
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This sequence can be transformed into geometric sum using the following manipulations:

- Divide both equalities by 2^n:

$$\begin{aligned} T_0/2^0 &= 0 \\ T_n/2^n &= T_{n-1}/2^{n-1} + 1/2^n \end{aligned}$$

- Set $S_n = T_n/2^n$ to have:

$$\begin{aligned} S_0 &= 0 \\ S_n &= S_{n-1} + 2^{-n} \end{aligned}$$

(This is geometric sum with the parameters $a = 1$ and $x = 1/2$.)

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$$\begin{aligned} T_0 &= 0 \\ T_n &= 2T_{n-1} + 1 \end{aligned}$$

Hence,

$$\begin{aligned} S_n &= \frac{0.5(0.5^n - 1)}{0.5 - 1} \text{ } (a_0 = 0 \text{ has been left out of the sum}) \\ &= 1 - 2^{-n} \end{aligned}$$

$$T_n = 2^n S_n = 2^n - 1$$

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$$T_n = 2^n S_n = 2^n - 1$$

Just the same result we have proven by means of induction! :))

Next subsection

Linear recurrence in form $a_n T_n = b_n T_{n-1} + c_n$

Here $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are arbitrary sequences and the initial value T_0 is a constant.

The idea:

Find a summation factor s_n satisfying the following property:

$$s_n b_n = s_{n-1} a_{n-1} \text{ for every } n \geqslant 1$$

If such a factor exists, one can do following transformations:

- $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$.
- Set $S_n = s_n a_n T_n$ and rewrite the equation as:

$$\begin{aligned} S_0 &= s_0 a_0 T_0 \\ S_n &= S_{n-1} + s_n c_n \end{aligned}$$

- This yields a closed formula (!) for solution:

$$T_n = \frac{1}{s_n a_n}\left(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k\right) = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

Finding a summation factor

Assuming that $b_n \neq 0$ for every n:

- Set $s_0 = 1$
- Compute the next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$
\begin{aligned}
s_1 &= \frac{a_0}{b_1} \\
s_2 &= \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2} \\
s_3 &= \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3} \\
&\cdots\cdots\cdots \\
s_n &= \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \dots a_{n-1}}{b_1 b_2 \dots b_n}
\end{aligned}
$$

(To be proved by induction!)

Example: application of summation factor

$a_n = c_n = 1$ and $b_n = 2$ gives the Hanoi Tower sequence:

Evaluate the summation factor:

$$s_n = \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \ldots a_{n-1}}{b_1 b_2 \ldots b_n} = \frac{1}{2^n}$$

The solution is:

$$T_n = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n (1 - 2^{-n}) = 2^n - 1$$