Sums

ITT9132 Concrete Mathematics
Lecture 4 – 18 February 2019

Chapter Two

- **Summation factors**
- **Manipulation of sums**
- **Multiple sums**
- **General methods**

TAL TECH

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Linear recurrence in form $a_n T_n = b_n T_{n-1} + c_n$

Here $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are arbitrary sequences and the initial value T_0 is a constant.

The idea:

Find a summation factor s_n satisfying the following property:

$$s_n b_n = s_{n-1} a_{n-1} \text{ for every } n \geqslant 1$$

If such a factor exists, one can do following transformations:

- $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$.
- Set $S_n = s_n a_n T_n$ and rewrite the equation as:

$$S_0 = s_0 a_0 T_0$$
$$S_n = S_{n-1} + s_n c_n$$

- This yields a closed formula (!) for solution:

$$T_n = \frac{1}{s_n a_n} \left(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k \right) = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right)$$

Finding a summation factor

Assuming that $b_n \neq 0$ for every n:

- Set $s_0 = 1$
- Compute the next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$s_1 = \frac{a_0}{b_1}$$
$$s_2 = \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2}$$
$$s_3 = \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3}$$
$$\dots\dots\dots$$
$$s_n = \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \dots a_{n-1}}{b_1 b_2 \dots b_n}$$

(To be proved by induction!)

Example: application of summation factor

$a_n = c_n = 1$ and $b_n = 2$ gives the Hanoi Tower sequence:

Evaluate the summation factor:

$$s_n = \frac{s_{n-1}\,a_{n-1}}{b_n} = \frac{a_0\,a_1\ldots a_{n-1}}{b_1\,b_2\ldots b_n} = \frac{1}{2^n}$$

The solution is:

$$T_n = \frac{1}{s_n a_n}\left(s_1\,b_1\,T_0 + \sum_{k=1}^{n} s_k\,c_k\right) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n(1-2^{-n}) = 2^n - 1$$

Yet Another Example: constant coefficients

Equation $Z_n = aZ_{n-1} + b$

Taking $a_n = 1$, $b_n = a$ and $c_n = b$:

- Evaluate summation factor:

$$s_n = \frac{s_{n-1}\,a_{n-1}}{b_n} = \frac{a_0\,a_1\ldots a_{n-1}}{b_1\,b_2\ldots b_n} = \frac{1}{a^n}$$

- The solution is:

$$\begin{aligned} Z_n &= \frac{1}{s_n a_n}\left(s_1\,b_1\,Z_0 + \sum_{k=1}^{n} s_k\,c_k\right) = a^n\left(Z_0 + b\sum_{k=1}^{n}\frac{1}{a^k}\right) \\ &= a^n Z_0 + b\left(1 + a + a^2 + \cdots + a^{n-1}\right) \\ &= a^n Z_0 + \frac{a^n - 1}{a - 1}\,b \end{aligned}$$

Yet Another Example: check up on results

Equation $Z_n = aZ_{n-1} + b$

$$
\begin{aligned}
Z_n &= aZ_{n-1} + b \\
&= a^2 Z_{n-2} + ab + b \\
&= a^3 Z_{n-3} + a^2 b + ab + b
\end{aligned}
$$

$\cdots\cdots$

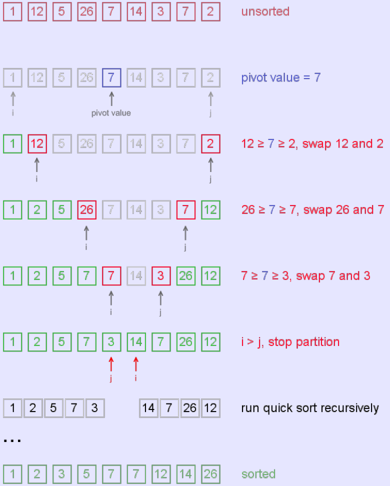
$$
\begin{aligned}
&= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b \\
&= a^k Z_{n-k} + \frac{a^k - 1}{a - 1} b \qquad \text{(assuming } a \neq 1\text{)}
\end{aligned}
$$

Continuing until $k = n$:

$$
\begin{aligned}
Z_n &= a^n Z_{n-n} + \frac{a^n - 1}{a - 1} b \\
&= a^n Z_0 + \frac{a^n - 1}{a - 1} b
\end{aligned}
$$

Efficiency of quick sort

Average number of comparisons: $C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k,\ C_0 = 0.$



Efficiency of quick sort (2)

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

and subtract to eliminate the sum:

$$\begin{aligned}
nC_n - (n-1)C_{n-1} &= n^2 + n + 2C_{n-1} - (n-1)^2 - (n-1) \\
nC_n - nC_{n-1} + C_{n-1} &= n^2 + n + 2C_{n-1} - n^2 + 2n - 1 - n + 1 \\
nC_n - nC_{n-1} &= C_{n-1} + 2n \\
nC_n &= (n+1)C_{n-1} + 2n
\end{aligned}$$

Efficiency of quick sort (3)

Equation $nC_n = (n+1)C_{n-1} + 2n$

- Evaluate summation factor with $a_n = n$, $b_n = n+1$ and $c_n = 2n$:

$$s_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n} = \frac{1 \cdot 2 \cdots (n-1)}{3 \cdot 4 \cdots (n+1)} = \frac{2}{n(n+1)}$$

- Then the solution of the recurrence is:

$$\begin{aligned} C_n &= \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right) \\ &= \frac{n+1}{2}\sum_{k=1}^{n}\frac{4k}{k(k+1)} \\ &= 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1} = 2(n+1)\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} - 1\right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n$ is the nth harmonic number.

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Manipulation of Sums

Some properties of sums:

For every finite set K and permutation $p(k)$ of K:

Distributive law

$$\sum_{k\in K} ca_k = c\sum_{k\in K} a_k$$

Associative law

$$\sum_{k\in K}(a_k+b_k) = \sum_{k\in K} a_k + \sum_{k\in K} b_k$$

Commutative law

$$\sum_{k\in K} a_k = \sum_{p(k)\in K} a_{p(k)}$$

Application of these laws for $S=\sum_{0\leqslant k\leqslant n}(a+bk)$

$$\begin{array}{lll} S & =\sum_{0\leqslant n-k\leqslant n}(a+b(n-k)) & =\sum_{0\leqslant k\leqslant n}(a+bn-bk) \\ 2S & =\sum_{0\leqslant k\leqslant n}((a+bk)+(a+bn-bk)) & =\sum_{0\leqslant k\leqslant n}(2a+bn) \\ 2S & =(2a+bn)\sum_{0\leqslant k\leqslant n}1 & =(2a+bn)(n+1) \end{array}$$

Hence, $S=(n+1)a+\frac{n(n+1)}{2}b.$

Yet Another Useful Equality

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cup K'} a_k + \sum_{k \in K \cap K'} a_k$$

Special cases:

a. For $1 \leqslant m \leqslant n$:

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k$$

b. For $n \geqslant 0$:

$$\sum_{0 \leqslant k \leqslant n} a_k = a_0 + \sum_{1 \leqslant k \leqslant n} a_k$$

c. For $n \geqslant 0$:

$$S_n + a_{n+1} = a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}$$

Example: $S_n = \sum_{k=0}^{n} kx^k$

- For $x \neq 1$:

$$\begin{aligned} S_n + (n+1)x^{n+1} &= \sum_{0 \leqslant k \leqslant n} (k+1)x^{k+1} \\ &= \sum_{0 \leqslant k \leqslant n} kx^{k+1} + \sum_{0 \leqslant k \leqslant n} x^{k+1} \\ &= xS_n + \frac{x(1-x^{n+1})}{1-x} \end{aligned}$$

- From this we get:

$$\sum_{k=0}^{n} kx^k = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(x-1)^2}$$

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Multiple sums

Definition

If K_1 and K_2 are index sets, then:

$$\sum_{i \in K_1, j \in K_2} a_{i,j} = \sum_i \left(\sum_j a_{i,j} \, [P(i,j)] \right)$$

where P is the predicate $P(i,j) = (i \in K_1) \wedge (j \in K_2)$.

Law of *interchange of the order of summation*:

$$\sum_j \sum_k a_{j,k} \, [P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} \, [P(j,k)]$$

If $a_{j,k} = a_j b_k$, then:

$$\sum_{j \in J, k \in K} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} b_k \right)$$

Multiple sums with independent indices

If $P(j,k) = Q(j) \wedge R(k)$, where Q and R are properties and $\wedge$ indicates the logical conjunction (AND), then the indices j and k are independent and the double sum can be rewritten:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}\,([Q(j) \wedge R(k)]) \\
&= \sum_{j,k} a_{j,k}[Q(j)][R(k)] \\
&= \sum_{j}[Q(j)] \sum_{k} a_{j,k} R(k) = \sum_{j}\sum_{k} a_{j,k} \\
&= \sum_{k} a_{j,k}[R(k)] \sum_{j}[Q(j)] = \sum_{k}\sum_{j} a_{j,k}
\end{aligned}$$

Multiple sums with dependent indices

In general, the indices are not independent, but we can write:

$$P(j,k) = Q(j) \wedge R'(j,k) = R(k) \wedge Q'(j,k)$$

In this case, we can proceed as follows:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}[Q(j)][R'(j,k)] \\
&= \sum_{j}[Q(j)] \sum_{k} a_{j,k}[R'(j,k)] = \sum_{j \in J} \sum_{k \in K'} a_{j,k} \\
&= \sum_{k}[R(k)] \sum_{j} a_{j,k}[Q'(j,k)] = \sum_{k \in K} \sum_{j \in J'} a_{j,k}
\end{aligned}$$

where:

- $J = \{j \mid Q(j)\}, K' = \{k \mid R'(j,k)\} = K'(j)$
- $K = \{k \mid R(k)\}, J' = \{j \mid Q'(j,k)\} = J'(k)$

Warmup: what's wrong with this sum?

$$\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k}$$

$$= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k}$$

$$= \sum_{k=1}^{n} \sum_{k=1}^{n} 1$$

$$= n^2$$

Warmup: what's wrong with this sum?

$$\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k}$$
$$= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k}$$
$$= \sum_{k=1}^{n} \sum_{k=1}^{n} 1$$
$$= n^2$$

Solution

The second passage is seriously wrong:
It is not licit to turn two independent variables into two dependent ones.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \le j \le n} \sum_{j \le k \le n} a_j a_k$.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \le j \le n} \sum_{j \le k \le n} a_j a_k$.

A crucial observation

$$[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]$$

Hence,

$$\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{k=1}^{n} \sum_{j=1}^{k} a_j a_k$$

Also,

$$[1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le j, k \le n] + [1 \le j = k \le n]$$

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \le j \le n} \sum_{j \le k \le n} a_j a_k$.

A crucial observation (cont.)

This can also be understood by considering the following matrix:

$$\begin{pmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 & \dots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & a_2 a_3 & \dots & a_2 a_n \\ a_3 a_1 & a_3 a_2 & a_3 a_3 & \dots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & a_n a_n \end{pmatrix}$$

and observing that $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = S_U$ is the sum of the elements of the upper triangular part of the matrix.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} a_j a_k$.

A crucial observation (end)

If we add to S_U the sum $S_L = \sum_{k=1}^{n} \sum_{j=1}^{k} a_j a_k$ of the elements of the lower triangular part of the matrix, we count each element of the matrix once, except those on the main diagonal, which we count twice.

But the matrix is symmetric, so $S_U = S_L$, and

$$S_U = \frac{1}{2} \left(\left(\sum_{k=1}^{n} a_k \right)^2 + \sum_{k=1}^{n} a_k^2 \right)$$

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Examples of multiple summation

Example 1

$$
\begin{aligned}
S_n &= \sum_{1\leqslant k\leqslant n}\ \sum_{1\leqslant j<k} \frac{1}{k-j} \\
&= \sum_{1\leqslant k\leqslant n}\ \sum_{1\leqslant k-j<k} \frac{1}{j} \\
&= \sum_{1\leqslant k\leqslant n}\ \sum_{0<j\leqslant k-1} \frac{1}{j} \\
&= \sum_{1\leqslant k\leqslant n} H_{k-1} \\
&= \sum_{1\leqslant k+1\leqslant n} H_k \\
&= \sum_{0\leqslant k<n} H_k
\end{aligned}
$$

Examples of multiple summation

Example 2

$$
\begin{aligned}
S_n &= \sum_{1 \leqslant j \leqslant n} \sum_{j < k \leqslant n} \frac{1}{k - j} \\
&= \sum_{1 \leqslant j \leqslant n} \sum_{j < k + j \leqslant n} \frac{1}{k} \\
&= \sum_{1 \leqslant j \leqslant n} \sum_{0 < k \leqslant n - j} \frac{1}{k} \\
&= \sum_{1 \leqslant j \leqslant n} H_{n-j} \\
&= \sum_{1 \leqslant n - j \leqslant n} H_j \\
&= \sum_{0 \leqslant j < n} H_j
\end{aligned}
$$

Examples of multiple summation

Example 3

$$
\begin{aligned}
S_n &= \sum_{1 \leqslant j < k \leqslant n} \frac{1}{k-j} \\
&= \sum_{1 \leqslant j < k+j \leqslant n} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j \leqslant n-k} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n-k}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n}{k} - \sum_{1 \leqslant k \leqslant n} 1 \\
&= n \left(\sum_{1 \leqslant k \leqslant n} \frac{1}{k} \right) - n \\
&= nH_n - n
\end{aligned}
$$

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General Methods: a Review

Example

$$\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \quad \text{for } n \geqslant 0$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
$\Box_n$	0	1	5	14	30	55	91	140	204	285	385	506	650

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Review: Method 0

Example: $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \text{ for } n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (http://oeis.org/)
- *etc*

Possible answer:

$$\Box_n = \frac{n(n+1)(2n+1)}{6} \qquad \text{for } n \geqslant 0$$

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Review: Method 1

Example: $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90
$3\Box_n/n(n+1)$	–	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

Review: Method 1

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\square_n$	0	1	5	14	30	55	91	140	204	285
$\square_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\square_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90
$3\square_n/n(n+1)$	–	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

Hypothesis:

$$\frac{3\square_n}{n(n+1)} = n + \frac{1}{2} \implies \square_n = \frac{n(n+1/2)(n+1)}{3} = \frac{n(n+1)(2n+1)}{6}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

The formula is trivially true for $n = 0$

Assume that the formula is true for $n-1$.

We know that $\square_n = \square_{n-1} + n^2$

We have

$$
\begin{aligned}
3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\
&= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\
&= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
&= n(n+\frac{1}{2})(n+1)
\end{aligned}
$$

Q.E.D.

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Review: Method 2

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum: $\cube_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have:

$$\cube_n + (n+1)^3 = \sum_{0 \leqslant k \leqslant n} (k+1)^3 = \sum_{0 \leqslant k \leqslant n} (k^3 + 3k^2 + 3k + 1)$$
$$= \cube_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Delete $\cube_n$ and extract $\square_n$:

$$3\square_n = (n+1)^3 - 3\frac{(n+1)n}{2} - (n+1)$$
$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$
$$= (n+1)(n+\frac{1}{2})n$$

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Review: Method 3

Example: $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence: $R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$ with $R_0 = 0$.
We look for a solution $R_n = \alpha A(n) + \beta B(n) + \gamma C(n)$ for suitable $A(n)$, $B(n)$, $C(n)$.

Review: Method 3

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence: $R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$ with $R_0 = 0$.
We look for a solution $R_n = \alpha A(n) + \beta B(n) + \gamma C(n)$ for suitable $A(n)$, $B(n)$, $C(n)$.

Case 1: $R_n = n$

- Equation: $n = n - 1 + \alpha + \beta n + \gamma n^2$ for every $n \geqslant 1$.
- Then $\alpha = 1$ and $\beta = \gamma = 0$, and $A(n) = n$.

Review: Method 3

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence: $R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$ with $R_0 = 0$.
We look for a solution $R_n = \alpha A(n) + \beta B(n) + \gamma C(n)$ for suitable $A(n)$, $B(n)$, $C(n)$.

Case 2: $R_n = n^2$

- Equation: $n^2 = (n-1)^2 + \alpha + \beta n + \gamma n^2$ for every $n \geqslant 1$;
- or: $0 = (\alpha + 1) + (\beta - 2)n + \gamma n^2$.
- Then $\alpha = -1$, $\beta = 2$, $\gamma = 1$: we obtain the equation $-A(n) + 2B(n) = n^2$
- As $A(n) = n$, we find $B(n) = \frac{n(n+1)}{2}$

Review: Method 3

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence: $R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$ with $R_0 = 0$.
We look for a solution $R_n = \alpha A(n) + \beta B(n) + \gamma C(n)$ for suitable $A(n)$, $B(n)$, $C(n)$.

Case 3: $R_n = n^3$

- Equation: $n^3 = (n-1)^3 + \alpha + \beta n + \gamma n^2$
- or: $(\alpha - 1) + (\beta + 3)n + (\gamma - 3)n^2 = 0$.
- Then $\alpha = 1$, $\beta = -3$, $\gamma = 3$: we obtain the equation $A(n) - 3B(n) + 3C(n) = n^3$.
- As $A(n) = n$ and $B(n) = \frac{n(n+1)}{2}$, we find:

$$6C(n) = 2n^3 - 2n + 3(n^2 + n) = 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1) = n(2n+1)(n+1)$$

Review: Method 3

Example: $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence: $R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$ with $R_0 = 0$.
We look for a solution $R_n = \alpha A(n) + \beta B(n) + \gamma C(n)$ for suitable $A(n)$, $B(n)$, $C(n)$.

Summarizing:

- $R_n = \Box_n$ corresponds to $\alpha = \beta = 0$, $\gamma = 1$.
- The solution of the recurrence is thus:

$$\Box_n = C(n) = \frac{n(n+1)(2n+1)}{6}$$

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1 Sums and Recurrences
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3 Multiple sums

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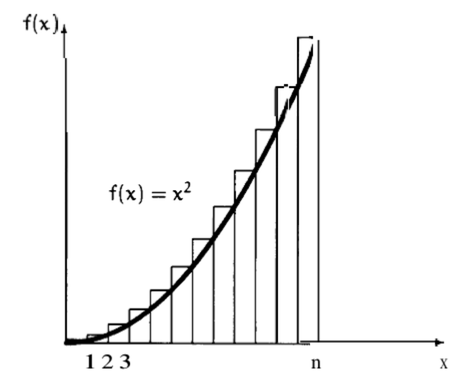
Review: Method 4

Example: $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.



$$\int_0^n x^2\,dx = \frac{n^3}{3} \tag{1}$$

$$\Box_n = \int_0^n x^2\,dx + E_n \tag{2}$$

$$E_n = \sum_{k=1}^{n}\left(k^2 - \int_{k-1}^{k} x^2\,dx\right) \tag{3}$$

Review: Method 4

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.

Evaluate (3):

$$
\begin{aligned}
E_n &= \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \, dx \right) \\
&= \sum_{k=1}^{n} \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\
&= \sum_{k=1}^{n} \left(k - \frac{1}{3} \right) \\
&= \frac{(n+1)n}{2} - \frac{n}{3} = \frac{3n^2 + n}{6} .
\end{aligned}
$$

Finally, from (2) and (1) we get :

$$\square_n = \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

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Review: Method 5

Example: $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Expand and Contract.

$$
\begin{aligned}
\Box_n &= \sum_{1 \leqslant k \leqslant n} k^2 = \sum_{1 \leqslant k \leqslant n} \left(\sum_{1 \leqslant j \leqslant k} 1 \right) k = \sum_{1 \leqslant j \leqslant k \leqslant n} k \\
&= \sum_{1 \leqslant j \leqslant n} \sum_{j \leqslant k \leqslant n} k = \sum_{1 \leqslant j \leqslant n} \left(\sum_{1 \leqslant k \leqslant n} k - \sum_{1 \leqslant k < j} k \right) \\
&= \sum_{1 \leqslant j \leqslant n} \left(\frac{n(n+1)}{2} - \frac{(j-1)j}{2} \right) \\
&= \frac{1}{2} \left(n^2(n+1) - \sum_{1 \leqslant j \leq n} j^2 + \sum_{1 \leqslant j \leqslant n} j \right) \\
&= \frac{1}{2} n^2(n+1) - \frac{1}{2} \Box_n + \frac{1}{4} n(n+1)
\end{aligned}
$$

Hence,

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TAL TECH

Review: Other methods

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

- Finite calculus
- Generating functions