Sums

ITT9132 Concrete Mathematics
Lecture 5 – 25 February 2019

Chapter Two

- **Finite and Infinite Calculus**
- **Derivative and Difference Operators**
- **Integrals and Sums**
- **Summation by Parts**
- **Infinite Sums**

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Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{dy}{dx} = \frac{df}{dx}(x) = \frac{df(x)}{dx} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

In general, if $h \in \mathbb{R}$ (or $h \in \mathbb{C}$), then

Forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

Backward difference

$$\nabla_h[f](x) = f(x) - f(x-h)$$

Central difference

$$\delta_h[f](x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h)$$

$$\mathrm{D}f(x) = \lim_{h\to 0} \frac{\Delta_h[f](x)}{h}$$

Derivative of Power function

Example: $f(x) = x^3$

In this case,

$$\begin{aligned}\Delta_h[f](x) &= f(x+h) - f(x)\\ &= (x+h)^3 - x^3\\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3\\ &= h\cdot(3x^2 + 3xh + h^2)\end{aligned}$$

Hence,

$$\mathrm{D}f(x) = \lim_{h\to 0}\frac{h\cdot(3x^2+3xh+h^2)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

In general, for $m \geq 1$ integer:

$$\mathrm{D}(x^m) = mx^{m-1}$$

(Forward) Difference of Power Function

Example: $f(x) = x^3$

In this case,

$$\Delta f(x) = \Delta_1 [f](x) = 3x^2 + 3x + 1$$

In general, for $m \geq 1$ integer:

$$\Delta(x^m) = \sum_{k=1}^{m} \binom{m}{k} x^{m-k}$$

because of Newton's binomial theorem.

Falling and Rising Factorials

Definition

The falling factorial (power) is defined for $m \geqslant 0$ by:

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

The rising factorial (power) is defined for $m \geqslant 0$ by:

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

Properties

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}}$$

$$n! = n^{\underline{n}} = 1^{\overline{n}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

$$x^{\underline{m}} = \frac{x^{\underline{m+1}}}{x-m} \quad \text{if } x \neq m$$

$$x^{\underline{-m}} = \frac{1}{(x+1)^{\overline{m}}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$

Difference of falling factorial with positive exponent

$$\begin{aligned}\Delta\left(x^{\underline{m}}\right) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\ &= (x+1)\cdot(x\cdots(x-m+2)) - (x\cdots(x-m+2))\cdot(x-m+1) \\ &= (x+1-(x-m+1))\cdot(x\cdots(x-m+2)) \\ &= m\cdot x^{\underline{m-1}}\end{aligned}$$

Hence:

$$\Delta(x^{\underline{m}}) = mx^{\underline{m-1}} \ \forall m \geqslant 1$$

Difference of falling factorial with negative exponent: Example

Let's check this formula for negative power:

$$
\begin{aligned}
\Delta x^{\underline{-2}} &= (x+1)^{\underline{-2}} - x^{\underline{-2}} \\
&= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\
&= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)} \\
&= \frac{-2}{(x+1)(x+2)(x+3)} \\
&= -2 \cdot x^{\underline{-3}}
\end{aligned}
$$

Difference of falling factorial with negative exponent

$$
\begin{aligned}
\Delta x^{\underline{-m}} &= (x+1)^{\underline{-m}} - x^{\underline{-m}} \\
&= \frac{1}{(x+2)\cdots(x+m)(x+m+1)} - \frac{1}{(x+1)(x+2)\cdots(x+m)} \\
&= \frac{(x+1)-(x+m+1)}{(x+1)(x+2)\cdots(x+m)(x+m+1)} \\
&= \frac{-m}{(x+1)(x+2)\cdots(x+m)(x+m+1)} \\
&= -mx^{\underline{-(m+\mathbf{1})}} \\
&= -mx^{\underline{-m-\mathbf{1}}}
\end{aligned}
$$

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Indefinite Integrals and Sums

The Fundamental Theorem of Calculus

$$\mathrm{D}f(x) = g(x) \quad \text{iff} \quad \int g(x)\mathrm{d}x = f(x) + \mathrm{C}$$

Definition

The indefinite sum of the function $g(x)$ is the class of functions f such that $\Delta f(x) = g(x)$:

$$\Delta f(x) = g(x) \quad \text{iff} \quad \sum g(x)\delta x = f(x) + C(x)$$

where $C(x)$ is a function such that $C(x+1) = C(x)$ for any integer value of x.

Definite Integrals and Sums

If $g(x) = \mathrm{D}f(x)$, then:

$$\int_a^b g(x)\mathrm{d}x = f(x)\Big|_a^b = f(b) - f(a)$$

Similarly:

If $g(x) = \Delta f(x)$, then:

$$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a)$$

Definite sums

Some observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x = f(b+1) - f(b) = g(b)$

Hence,

$$\begin{aligned}
\sum_a^b g(x)\delta x &= \sum_{k=a}^{b-1} g(k) = \sum_{a\leqslant k<b} g(k) \\
&= \sum_{a\leqslant k<b} (f(k+1) - f(k)) \\
&= (f(a+1) - f(a)) + (f(a+2) - f(a+1)) + \cdots \\
&+ (f(b-1) - f(b-2)) + (f(b) - f(b-1)) \\
&= f(b) - f(a)
\end{aligned}$$

Integrals and Sums of Powers

If $m \neq -1$, then:

$$\int_0^n x^m \mathrm{d}x = \frac{x^{m+1}}{m+1}\Big|_0^n = \frac{n^{m+1}}{m+1}$$

Analogous finite case:

If $m \neq -1$, then:

$$\sum_0^n k^{\underline{m}} \delta x = \sum_{0 \leqslant k < n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

Sums of Powers: applications

Case $m = 1$

$$\sum_{0 \leqslant k < n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

Case $m = 2$ Due to $k^2 = k^{\underline{2}} + k^{\underline{1}}$ we get:

$$\begin{aligned}
\sum_{0 \leqslant k < n} k^2 &= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\
&= \frac{1}{3} n(n-1)(n-2) + \frac{1}{2} n(n-1) \\
&= \frac{1}{6} n\left(2(n-1)(n-2) + 3(n-1)\right) \\
&= \frac{1}{6} n(n-1)(2n-4+3) \\
&= \frac{1}{6} n(n-1)(2n-1)
\end{aligned}$$

Taking $n+1$ instead of n gives:

$$\Box_n = \frac{(n+1)n(2n+1)}{6}$$

Sums of Powers (case $m = -1$)

As a first step, we observe that:

$$\begin{aligned}\Delta H_x &= H_{x+1} - H_x \\ &= \left(1 + \frac{1}{2} + \ldots + \frac{1}{x} + \frac{1}{x+1}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{x}\right) \\ &= \frac{1}{x+1} = x^{\underline{-1}}\end{aligned}$$

We conclude:

$$\sum_a^b x^{\underline{-1}}\,\delta x = H_x\Big|_a^b$$

Sums of Discrete Exponential Functions

- We have:

$$\mathrm{D}e^x = e^x$$

The finite analogue should have $\Delta f(x) = f(x)$. This means:

$$f(x+1) - f(x) = f(x), \text{ that is, } f(x+1) = 2f(x), \text{ only possible if } f(x) = 2^x$$

- For general base $c > 0$, the difference of c^x is:

$$\Delta(c^x) = c^{x+1} - c^x = (c-1)c^x$$

and the "anti-difference" for $c \neq 1$ is $\dfrac{c^x}{(c-1)}$.

As an application, we compute the sum of the geometric progression:

$$\sum_{a \leqslant k < b} c^k = \sum_a^b c^x \delta x = \frac{c^x}{c-1}\Big|_a^b = \frac{c^b - c^a}{c-1} = c^a \cdot \frac{c^{b-a} - 1}{c-1}.$$

Differential equations and difference equations

Differential equation	Solution	Difference equation	Solution
$Df_n(x) = nf_{n-1}(x)$ $f_n(0) = [n=0]\,,\ n \geqslant 0$	$f_n(x) = x^n$	$\Delta u_m(x) = mu_{m-1}(x)$ $u_m(0) = [m=0]\,,\ m \geqslant 0$	$u_m(x) = x^{\underline{m}}$
$Df_n(x) = nf_{n-1}(x)$ $f_n(1) = 1\,,\ n < 0$	$f_n(x) = x^n$	$\Delta u_m(x) = mu_{m-1}(x)$ $u_m(0) = 1/m!\,,\ m < 0$	$u_m(x) = x^{\underline{m}}$
$Df(x) = \frac{1}{x} \cdot [x > 0]$ $f(1) = 1$	$f(x) = \ln x$	$\Delta u(x) = \frac{1}{x} \cdot [x \geqslant 1]$ $u(1) = 1$	$u(x) = H_x$
$Df(x) = f(x)$ $f(0) = 1$	$f(x) = e^x$	$\Delta u(x) = u(x)$ $u(0) = 1$	$u(x) = 2^x$
$Df(x) = b \cdot f(x)$ $f(0) = 1$	$f(x) = a^x$ where $b = \ln a$	$\Delta u(x) = b \cdot u(x)$ $u(0) = 1$	$u(x) = c^x$ where $b = c - 1$

l'Hôpital's rule and Stolz-Cesàro lemma

l'Hôpital's rule: Hypotheses

1. $f(x)$ and $g(x)$ are both vanishing or both infinite at x_0.
2. $g'(x)$ is always positive in some neighborhood of x_0.

l'Hôpital's rule: Thesis

- If $\lim_{x\to x_0} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$,
- then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = L$.

Stolz-Cesàro lemma: Hypotheses

1. $u(n)$ and $v(n)$ are defined for every value $n \in \mathbb{N}$.
2. $v(n)$ is positive, strictly increasing, and divergent.

Stolz-Cesàro lemma: Thesis

- If $\lim_{n\to\infty} \dfrac{\Delta u(n)}{\Delta v(n)} = L \in \mathbb{R}$,
- then $\lim_{n\to\infty} \dfrac{u(x)}{v(x)} = L$.

Next subsection

Summation by Parts

Infinite analogue: integration by parts

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

Difference of a product

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)\\ &= \Delta u(x)v(x+1) + u(x)\Delta v(x)\\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x)\end{aligned}$$

where E is the shift operator $Ef(x) = f(x+1)$. We then have the:

Rule for summation by parts

$$\sum u\Delta v\,\delta x = uv - \sum Ev\Delta u\,\delta x$$

Why the shift?

If we repeat our derivation with two continuous functions f and g of one real variable x, we find for any increment $h \neq 0$:

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x) \\ &= f(x)(g(x+h) - g(x)) + g(x+h)(f(x+h) - f(x)) \end{aligned}$$

The incremental ratio is thus:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x) \cdot \frac{g(x+h) - g(x)}{h} + g(x+h) \cdot \frac{f(x+h) - f(x)}{h}$$

So there is a shift: but it is infinitesimal—and disappears by continuity of g.

Summation by Parts (2)

Example: $S = \sum_{k=0}^{n} k2^k$

- Taking $u(x) = x$, $v(x) = 2^x$ and $Ev(x) = 2^{x+1}$:

$$\sum x2^x \delta x = x2^x - \sum 2^{x+1} \delta x = x2^x - 2^{x+1} + C$$

This yields:

$$\begin{aligned}
\sum_{k=0}^{n} k2^k &= \sum_{0}^{n+1} x2^x \delta x \\
&= \left(x2^x - 2^{x+1}\right)\Big|_0^{n+1} \\
&= \left((n+1)2^{n+1} - 2^{n+2}\right) - (0 \cdot 2^0 - 2) \\
&= (n-1)2^{n+1} + 2
\end{aligned}$$

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Summation by Parts (3)

Example: $S = \sum_{k=0}^{n-1} kH_k$

Continuous analogue:

$$\begin{aligned}
\int x \ln x \,\mathrm{d}x &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x \\
&= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,\mathrm{d}x \\
&= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} \\
&= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)
\end{aligned}$$

Summation by Parts (3)

Example: $S = \sum_{k=0}^{n-1} kH_k$

- Taking $u(x) = H_x$ and $vx = x^{\underline{2}}/2$, we get $\Delta u(x) = \Delta H_x = x^{\underline{-1}} \frac{1}{x+1}$, $\Delta v(x) = x = x^{\underline{1}}$, and $Ev(x) = \frac{(x+1)^{\underline{2}}}{2}$. Then:

$$\sum_{k=0}^{n-1} kH_k = \sum_0^n xH_x \,\delta x = uv\Big|_0^n - \sum_0^n Ev\Delta u \,\delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x\Big|_0^n - \sum_0^n \frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}} \,\delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x\Big|_0^n - \frac{1}{2}\sum_0^n x \,\delta x$$

$$= \left(\frac{x^{\underline{2}}}{2} H_x - \frac{1}{2}\cdot\frac{x^{\underline{2}}}{2}\right)\Big|_0^n$$

$$= \frac{n^{\underline{2}}}{2}\left(H_n - \frac{1}{2}\right)$$

Next section

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How to sum infinite number sequences?

Setting $\sum_{k\in\mathbb{N}} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ seems meaningful ...

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots.$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

How to sum infinite number sequences?

Setting $\sum_{k\in\mathbb{N}} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ seems meaningful ...

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

But can we manipulate such sums like we do with finite sums?

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How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \ldots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \ldots = T - 1$$

and

$$T = -1$$

How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \ldots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \ldots = T - 1$$

and

$$T = -1$$

Problem:

- The sum T is infinite . . .
- and we cannot subtract an infinite quantity from another infinite quantity.

How to sum infinite number sequences?

Example 3

Let

$$\sum_{k \geqslant 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Different ways to sum

$$(1-1) + (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + 0 + \dots = 0$$

and

$$1 - (1-1) - (1-1) - (1-1) - (1-1) - \dots = 1 - 0 - 0 - 0 - 0 - 0 - \dots = 1$$

How to sum infinite number sequences?

Example 3

Let

$$\sum_{k \geqslant 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Different ways to sum

$$(1-1) + (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + 0 + \dots = 0$$

and

$$1 - (1-1) - (1-1) - (1-1) - (1-1) - \dots = 1 - 0 - 0 - 0 - 0 - 0 - \dots = 1$$

Problem:

- The sequence of the partial sums does not converge ...
- and we cannot manipulate something that does not exist.

Defining Infinite Sums: Nonnegative Summands

Definition 1

If $a_k \geqslant 0$ for every $k \geqslant 0$, then:

$$\sum_{k \geqslant 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k = \sup_{K \subseteq \mathbb{N}, |K| < \infty} \sum_{k \in K} a_k$$

Note that:

- The definition as a limit is (sort of) a *Riemann integral*.
- The definition as a least upper bound is a *Lebesgue integral*.
- The limit / least upper bound above can be finite or infinite, but are always equal.
  Exercise: Prove this fact.

Defining Infinite Sums: Riemann Summation

Definition 2 (Riemann sum of a series)

A series $\sum_{k \geqslant 0} a_k$ with real coefficients converges to a real number S, called the sum of the series, if:

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k = S \,.$$

In this case, we write: $\sum_{k \geqslant 0} a_k = S$.
The values $S_n = \sum_{k=0}^{n} a_k$ are called the partial sums of the series.
The series $\sum_{k \geqslant 0} a_k$ converges absolutely if $\sum_{k \geqslant 0} |a_k|$ converges.

If the series $\sum_{k \geqslant 0} a_k = \sum_{k \geqslant 0} (b_k + i c_k)$ has complex coefficients, we say that it converges to $S = T + iU$ if $\sum_{k \geqslant 0} b_k$ converges to T and $\sum_{k \geqslant 0} c_k$ converges to U.

A series that converges, but not absolutely

Let $a_k = \dfrac{(-1)^{k-1}}{k} [k > 0]$. Then $\sum_{k \geqslant 0} a_k = \ln 2$.
However, it is easy to prove by induction that $\sum_{k=0}^{2^n} |a_k| = H_{2^n} > \dfrac{n}{2}$ for every $n \geqslant 1$.

Infinite Sums: Associativity

Associativity

A series $\sum_{k\geqslant 0} a_k$ has the associative property if for every two *strictly increasing* sequences

$$i_0 = 0 < i_1 < i_2 < \ldots < i_k < i_{k+1} < \ldots$$
$$j_0 = 0 < j_1 < j_2 < \ldots < j_k < j_{k+1} < \ldots$$

we have:

$$\sum_{k\geqslant 0} \left(\sum_{i=i_k}^{i_{k+1}-1} a_i \right) = \sum_{k\geqslant 0} \left(\sum_{j=j_k}^{j_{k+1}-1} a_i \right)$$

We have seen that the series $\sum_{k\geqslant 0} (-1)^k$ *does not* have the associative property.

Theorem

A series has the associative property if and only if it is convergent.

Proof: Regrouping as in the definition means *taking a subsequence* of the sequence of partial sums, which can converge to any of the latter's limit point.

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Defining Infinite Sums: Lebesgue Summation

Every real number can be written as $x = x^+ - x^-$, where:

$$x^+ = x \cdot [x > 0] = \max(x, 0) \text{ and } x^- = -x \cdot [x < 0] = \max(-x, 0)$$

Note that: $x^+ \geqslant 0$, $x^- \geqslant 0$, and $x^+ + x^- = |x|$.

Definition 3 (Lebesgue sum of a series)

Let $\{a_k\}_k$ be an absolutely convergent sequence of real numbers. Then:

$$\sum_k a_k = \sum_k a_k^+ - \sum_k a_k^-$$

The series $\sum_k a_k$:

- converges absolutely if $\sum_k a_k^+ < +\infty$ and $\sum_k a_k^- < +\infty$;
- diverges positively if $\sum_k a_k^+ = +\infty$ and $\sum_k a_k^- < +\infty$;
- diverges negatively if $\sum_k a_k^+ < +\infty$ and $\sum_k a_k^- = +\infty$.

If both $\sum_k a_k^+ = +\infty$ and $\sum_k a_k^- = +\infty$ then “Bad Stuff happens”.

Infinite Sums: Bad Stuff

Riemann series theorem

Let $\sum_k a_k$ be a series with real coefficients which converges, but not absolutely.
For every real number L there exists a permutation p of $\mathbb{N}$ such that:

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{p(k)} = L$$

Example: The harmonic series

If we rearrange the terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ as follows:

$$\begin{aligned} 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots &= \ldots + \frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k} + \ldots \\ &= \ldots + \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \ldots \end{aligned}$$

we obtain:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2 \quad \text{but} \quad 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2}\ln 2$$

Infinite Sums: Commutativity

Commutativity

A series $\sum_{k \geqslant 0} a_k$ has the commutative property if for every permutation p of $\mathbb{N}$,

$$\sum_{k \geqslant 0} a_{p(k)} = \sum_{k} a_k$$

The Riemann series theorem says that any series which is convergent, but not absolutely convergent, does not have the commutative property.

Theorem

A convergent series has the commutative property if and only if it is absolutely convergent.

Proof: (Sketch) Think of Lebesgue summation.

Infinite Sums: Commutativity

Commutativity

A series $\sum_{k \geqslant 0} a_k$ has the commutative property if for every permutation p of $\mathbb{N}$,

$$\sum_{k \geqslant 0} a_{p(k)} = \sum_{k} a_k$$

The Riemann series theorem says that any series which is convergent, but not absolutely convergent, does not have the commutative property.

Theorem

A convergent series has the commutative property if and only if it is absolutely convergent.

If we want to manipulate infinite sums like finite ones,
we must require absolute convergence.

Multiple infinite sums

Definition: Double infinite sums

For every $j, k \geqslant 0$ let $a_{j,k} \geqslant 0$.

1. If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j,k} a_{j,k} = \sup_{K \subseteq \mathbb{N} \times \mathbb{N}, |K| < \infty} \sum_{K} a_{j,k} = \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} \, .$$

   (Recall that $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = \sum_{j,k} a_{j,k} \, [0 \leqslant j \leqslant n] \, [0 \leqslant k \leqslant n]$.)

2. If $\sum_{j,k} |a_{j,k}| < +\infty$, then:

$$\sum_{j,k} a_{j,k} = \sum_{j,k} a^{+}_{j,k} - \sum_{j,k} a^{-}_{j,k} \, .$$

Multiple infinite sums

Definition: Double infinite sums

For every $j, k \geqslant 0$ let $a_{j,k} \geqslant 0$.

1. If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j,k} a_{j,k} = \sup_{K \subseteq \mathbb{N} \times \mathbb{N}, |K| < \infty} \sum_K a_{j,k} = \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} \,.$$

   (Recall that $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = \sum_{j,k} a_{j,k} [0 \leqslant j \leqslant n] [0 \leqslant k \leqslant n]$.)

2. If $\sum_{j,k} |a_{j,k}| < +\infty$, then:

$$\sum_{j,k} a_{j,k} = \sum_{j,k} a^+_{j,k} - \sum_{j,k} a^-_{j,k} \,.$$

Can we use $\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k}$ or $\sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k}$ instead?

Multiple infinite sums

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Can we use $\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k}$ or $\sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k}$ instead? In general, no:

- One writing is the limit on j of a limit on k which is a function of j;
- The other writing is the limit on k of a limit on j which is a function of k.
- There are no guarantees that the double limits be equal!

Multiple sums: An example of noncommutativity

From Joel Feldman's notes[1]

Let $a_{j,k} = [j = k = 0] + [k = j+1] - [k = j-1]$:

	0	1	2	3	4	...
0	1	1	0	0	0	...
1	−1	0	1	0	0	...
2	0	−1	0	1	0	...
3	0	0	−1	0	1	...
⋮	⋮	⋮	⋮	⋮	⋮	

Then:

- for every $j \geq 0$, $\sum_{k \geqslant 0} a_{j,k} = 2 \cdot [j = 0]$;
- for every $k \geq 0$, $\sum_{j \geqslant 0} a_{j,k} = 0$; and
- for every $n \geq 0$, $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = 1$.

Hence:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = 2 \; ; \quad \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = 0 \; ; \quad \text{and} \; \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} = 1 \, .$$

[1] http://www.math.ubc.ca/~feldman/m321/twosum.pdf retrieved 21.02.2019.

Multiple infinite sums: Swapping indices

Theorem

For $j, k \geqslant 0$ let $a_{j,k}$ be real numbers.

Tonelli If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,,$$

regardless of the quantities above being finite or infinite.

Fubini If $\sum_{j,k} |a_{j,k}| < +\infty$, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,.$$

Fubini's theorem is proved in the textbook.

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Fubini's theorem is proved in the textbook. Again:

If we want to manipulate infinite sums like finite ones,
we must require absolute convergence.

Cesàro summation

Given a series $\sum_k a_k$, consider the sequence $S_n = \sum_{k=0}^{n} a_k$ of the partial sums.

- Put $u(x) = \sum_{k=0}^{x-1} S_k$ and $v(x) = x$. Then $\Delta u(x) = S_x$ and $\Delta v(x) = 1$.
- Suppose $\sum_k a_k$ converges. Put $L = \sum_{k \geqslant 0} a_k = \lim_{n \to \infty} \frac{S_n}{1}$.
- We then have by the Stolz-Cesàro lemma: $\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} S_k}{n} = L$.

Given a (not necessarily convergent) series $\sum_k a_k$, the quantity:

$$C\sum_k a_k = \lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} S_k}{n}$$

is called the Cesàro sum of the series $\sum_k a_k$.

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A series can have a Cesàro sum without being convergent. For example, if $a_k = (-1)^k$, then $\sum_{k=0}^{n-1} S_k = \frac{n + [n \text{ is odd}]}{2}$, so $C\sum_k (-1)^k = \frac{1}{2}$.

n	0	1	2	3	4	5	6	7	8	9
a_n	1	−1	1	−1	1	−1	1	−1	1	−1
S_n	1	0	1	0	1	0	1	0	1	0
$\sum_{k=0}^{n-1} S_k$	0	1	1	2	2	3	3	4	4	5