Integer Functions

ITT9132 Concrete Mathematics
Lecture 6 – 1 March 2019

Contents

TAL TECH

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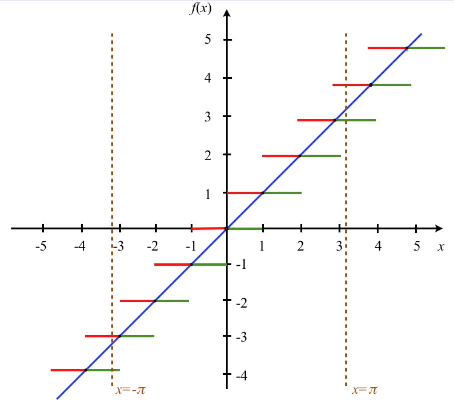
Floors and Ceilings

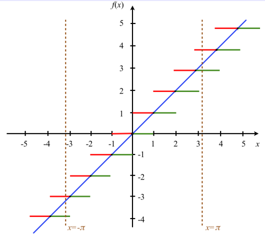
Definition

- The floor $\lfloor x \rfloor$ is the greatest integer not larger than x;
- The ceiling $\lceil x \rceil$ is the smallest integer not smaller than x.



$$\lfloor \pi \rfloor = 3 \quad \lfloor -\pi \rfloor = -4$$
$$\lceil \pi \rceil = 4 \quad \lceil -\pi \rceil = -3$$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$



For every $x \in \mathbb{R}$:

① $\lfloor x \rfloor = x = \lceil x \rceil$ iff $x \in \mathbb{Z}$

② $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$

③ $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$

④ $\lceil x \rceil - \lfloor x \rfloor = [x \notin \mathbb{Z}]$

Warmup: Representing numbers

Problem

Let $n = 2^m + \ell$. What are closed formulas for m and ℓ?

Warmup: Representing numbers

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Solution

First, $2^m \leqslant n < 2^{m+1}$.

- As lg, the logarithm in base 2, is an increasing function, $m \leqslant \lg n < m+1$.
- Then:

$$m = \lfloor \lg n \rfloor .$$

Next, $\ell = n - 2^m$. Then:

$$\ell = n - 2^{\lfloor \lg n \rfloor} .$$

Warmup: the generalized Dirichlet box principle

Statement of the principle

Let m and n be positive integers. If n items are stored into m boxes, then:

- at least one box will contain at least $\lceil n/m \rceil$ objects, and
- at least one box will contain at most $\lfloor n/m \rfloor$ objects.

Warmup: the generalized Dirichlet box principle

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Let m and n be positive integers. If n items are stored into m boxes, then:

- at least one box will contain at least $\lceil n/m \rceil$ objects, and
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Proof

By contradiction, assume each of the m boxes contains fewer than $\lceil n/m \rceil$ objects. Then

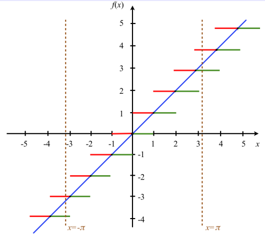
$$n \leqslant m \cdot \left(\left\lceil \frac{n}{m} \right\rceil - 1 \right) \text{ or equivalently}, \; \frac{n}{m} + 1 \leqslant \left\lceil \frac{n}{m} \right\rceil :$$

which is impossible.
Similarly, if each of the m boxes contained more than $\lfloor n/m \rfloor$ objects, we would have

$$n \geqslant m \cdot \left(\left\lfloor \frac{n}{m} \right\rfloor + 1 \right) \text{ or equivalently}, \; \frac{n}{m} - 1 \geqslant \left\lfloor \frac{n}{m} \right\rfloor :$$

which is also impossible.

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$ (cont.)



For every $x \in \mathbb{R}$ and $n \in \mathbb{Z}$:

⑤ $\lfloor x \rfloor = n$ iff $n \leqslant x < n+1$

⑥ $\lfloor x \rfloor = n$ iff $x-1 < n \leqslant x$

⑤ $\lceil x \rceil = n$ iff $n-1 < x \leqslant n$

⑧ $\lceil x \rceil = n$ iff $x \leqslant n < x+1$

⑨ $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ but, in general, $\lfloor nx \rfloor \neq n \lfloor x \rfloor$.

⑩ $\lceil x+n \rceil = \lceil x \rceil + n$ but, in general, $\lceil nx \rceil \neq n \lceil x \rceil$.

⑪ $x < n$ iff $\lfloor x \rfloor < n$

⑫ $n < x$ iff $n < \lceil x \rceil$

⑬ $x \leqslant n$ iff $\lceil x \rceil \leqslant n$

⑭ $n \leqslant x$ iff $n \leqslant \lfloor x \rfloor$

Generalization of property #9

Theorem

$$\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{if } 0 \leqslant \{x\} + \{y\} < 1\,, \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if } 1 \leqslant \{x\} + \{y\} < 2\,. \end{cases}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Proof. Let $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$. Then:

$$\begin{aligned} \lfloor x+y \rfloor &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor \end{aligned}$$

and clearly

$$\lfloor \{x\} + \{y\} \rfloor = \begin{cases} 0 & \text{if } 0 \leqslant \{x\} + \{y\} < 1\,, \\ 1 & \text{if } 1 \leqslant \{x\} + \{y\} < 2\,. \end{cases}$$

Q.E.D.

Warmup: When is $\lfloor nx \rfloor = n \lfloor x \rfloor$?

The problem

Give a necessary and sufficient condition on n and x so that

$$\lfloor nx \rfloor = n \lfloor x \rfloor$$

where n is a positive integer.

Warmup: When is $\lfloor nx \rfloor = n \lfloor x \rfloor$?

The problem

Give a necessary and sufficient condition on n and x so that

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where n is a positive integer.

The solution

Write $x = \lfloor x \rfloor + \{x\}$. Then

$$\lfloor nx \rfloor = \lfloor n \lfloor x \rfloor + n\{x\} \rfloor = n \lfloor x \rfloor + \lfloor n\{x\} \rfloor$$

As $\{x\}$ is nonnegative, so is $\lfloor n\{x\} \rfloor$. Then

$$\lfloor nx \rfloor = n \lfloor x \rfloor \text{ if and only if } \{x\} < 1/n$$

Next section

Floor/Ceiling Applications

Theorem

The binary representation of a natural number $n > 0$ has $m = \lfloor \log_2 n \rfloor + 1$ bits.

Proof.

$$n = \underbrace{a_{m-1} 2^{m-1} + a_{m-2} 2^{m-2} + \cdots + a_1 2 + a_0}_{m \text{ bits}} \text{ where } a_{m-1} = 1$$

Thus, $2^{m-1} \leqslant n < 2^m$, which gives $m - 1 \leqslant \log_2 n < m$. The last formula is valid if and only if $\lfloor \log_2 n \rfloor = m - 1$. Q.E.D.

Example: $n = 35 = 100011_2$

$$m = \lfloor \log_2 35 \rfloor + 1 = 5 + 1 = 6$$

Floor/Ceiling Applications

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Floor/Ceiling Applications (2)

Theorem

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing function with the property that, if $f(x) \in \mathbb{Z}$, then $x \in \mathbb{Z}$. Then:

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \quad \text{and} \quad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

whenever $f(x)$, $f(\lfloor x \rfloor)$, and $f(\lceil x \rceil)$ are all defined.

Proof. (for the ceiling function)

- If $x \in \mathbb{Z}$, then $x = \lceil x \rceil$, and there is nothing to prove.
- If $x \notin \mathbb{Z}$, then $x < \lceil x \rceil$, so $f(x) < f(\lceil x \rceil) \leqslant \lceil f(\lceil x \rceil) \rceil$ as f is strictly increasing. Also, $f(x) \leqslant \lceil f(x) \rceil \leqslant \lceil f(\lceil x \rceil) \rceil$ since the ceiling function is non-decreasing.
- If $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$, by the intermediate value theorem there exists y such that $x \leqslant y < \lceil x \rceil$ and $f(y) = \lceil f(x) \rceil$.
- Such y is an integer, because of f's special property, so actually $x < y < \lceil x \rceil$.
- But there are no integers strictly between x and $\lceil x \rceil$. This contradiction implies that we must have $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$.

Q.E.D.

Floor/Ceiling Applications (2a)

Example

- $\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$
- $\left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$
- $\left\lceil \frac{\left\lceil \frac{\lceil x \rceil / 10}{10} \right\rceil}{10} \right\rceil = \lceil x/1000 \rceil$
- $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$

In contrast:

$$\left\lceil \sqrt{\lfloor x \rfloor} \right\rceil \neq \lceil \sqrt{x} \rceil$$

For example, $\left\lceil \sqrt{\lfloor 1/4 \rfloor} \right\rceil = 0$ but $\left\lceil \sqrt{1/4} \right\rceil = 1$.

Floor/Ceiling Applications (3) : Intervals

For Real numbers $\alpha \neq \beta$

Range	Nr. of integer values of t	Restrictions
$\alpha \leqslant t \leqslant \beta$	$\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$	$\alpha \leqslant \beta$
$\alpha \leqslant t < \beta$	$\lceil \beta \rceil - \lceil \alpha \rceil$	$\alpha \leqslant \beta$
$\alpha < t \leqslant \beta$	$\lfloor \beta \rfloor - \lfloor \alpha \rfloor$	$\alpha \leqslant \beta$
$\alpha < t < \beta$	$\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$	$\alpha < \beta$

Floor/Ceiling Applications (3) : Intervals

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$\alpha < t \leqslant \beta$	$\lfloor \beta \rfloor - \lfloor \alpha \rfloor$	$\alpha \leqslant \beta$
$\alpha < t < \beta$	$\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$	$\alpha < \beta$

This is because, if $t \in \mathbb{Z}$, then:

$$\begin{array}{lll} \alpha \leqslant t & \text{if and only if} & \lceil \alpha \rceil \leqslant t \\ \alpha < t & \text{if and only if} & \lfloor \alpha \rfloor < t \ \text{ if and only if } \lfloor \alpha \rfloor + 1 \leqslant t \\ t \leqslant \beta & \text{if and only if} & t \leqslant \lfloor \beta \rfloor \\ t < \beta & \text{if and only if} & t < \lceil \beta \rceil \ \text{ if and only if } t \leqslant \lceil \beta \rceil - 1 \end{array}$$

and the slice $[m:n] = [m..n] \cap \mathbb{Z}$, $m \leqslant n$, has $n - m + 1$ elements.
(Note that, if $\alpha = \beta$ are both integers, then $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1 = -1$.)

A Game-Theoretical Application

The Concrete Mathematics Club Roulette

The Concrete Mathematics Casino[1] has a special roulette game:

- The roulette itself has 1000 slots, numbered from 1 to 1000.
- A number n is a winner if and only if $\lfloor \sqrt[3]{n} \rfloor$ is a factor of n.
- There is a bet of 1 dollar to play one round.
- If the number is a winner, players earn 5 dollars.

[1] Entrance reserved to book purchasers.

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- There is a bet of 1 dollar to play one round.
- If the number is a winner, players earn 5 dollars.

Is it convenient to play?

If there are W winning numbers and L losing numbers, then the average win is:

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6}{1000} W - 1$$

dollars, so:

$$\text{The game is convenient if and only if } W \geqslant \left\lceil \frac{1000}{6} \right\rceil = 167.$$

[1] Entrance reserved to book purchasers.

Winning or Losing at the Concrete Maths Roulette

We have:

$$
\begin{aligned}
W &= \sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n \right] \\
&= \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor \right] [k \mid n] [1 \leqslant n \leqslant 1000] \\
&= \sum_{k,m,n} \left[k^3 \leqslant n < (k+1)^3 \right] [n = km] [1 \leqslant n \leqslant 1000] \\
&= 1 + \sum_{k,m} \left[k^3 \leqslant km < (k+1)^3 \right] [1 \leqslant k < 10]
\end{aligned}
$$

because for $n = 1000$ it is only $k = 10, m = 100$

$$
\begin{aligned}
&= 1 + \sum_{k,m} \left[k^2 \leqslant m < (k+1)^3 / k \right] [1 \leqslant k < 10] \\
&= 1 + \sum_{1 \leqslant k < 10} \left(\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil \right) \\
&= 1 + \sum_{1 \leqslant k < 10} \left(3k + 3 + \lceil 1/k \rceil \right) \quad \text{but for } k \geqslant 1 \text{ it is } \lceil 1/k \rceil = 1 \\
&= 1 + 3 \cdot \frac{9 \cdot 10}{2} + 9 \cdot 4 = 1 + 135 + 36 = 172 > 167
\end{aligned}
$$

Floor/Ceiling Applications (3) : Spectra

Definition

The spectrum of a real number α is an infinite *multiset* of integers

$$\mathrm{Spec}(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\} = \{\lfloor n\alpha \rfloor \mid n \geqslant 1\}$$

An integer $m = \lfloor n\alpha \rfloor \in \mathrm{Spec}(\alpha)$ can appear for more than one value of n.
For example, $\mathrm{Spec}(1/2) = \{0,1,1,2,2,3,3,4,4,\ldots\}$.

Example.

$$\mathrm{Spec}(\sqrt{2}) = \{1,2,4,5,7,8,9,11,12,14,15,16,18,19,21,22,24,\ldots\}$$
$$\mathrm{Spec}(2+\sqrt{2}) = \{3,6,10,13,17,20,23,27,30,34,37,40,44,47,51,\ldots\}$$

Note that $\lfloor n(2+\sqrt{2}) \rfloor = \lfloor n\sqrt{2} \rfloor + 2n$.

Floor/Ceiling Applications (3) : Spectra

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For example, $\mathrm{Spec}(1/2) = \{0,1,1,2,2,3,3,4,4,\ldots\}$.

Theorem

If $\alpha, \beta \geqslant 1$ and $\alpha < \beta$ then $\mathrm{Spec}(\alpha) \neq \mathrm{Spec}(\beta)$.

Proof. As $\alpha, \beta \geqslant 1$, $\mathrm{Spec}(\alpha)$ and $\mathrm{Spec}(\beta)$ have no repetitions.

- Let $m \in \mathbb{Z}$ be so large that $m(\beta - \alpha) \geqslant 1$.
- For such m, $m\beta - m\alpha \geqslant 1$, hence $\lfloor m\beta\rfloor > \lfloor m\alpha\rfloor$.
- Thus $\mathrm{Spec}(\beta)$ has fewer than m elements which are $\leqslant \lfloor m\alpha\rfloor$, while $\mathrm{Spec}(\alpha)$ has at least m such elements. Q.E.D.

Spectra and partitions of integers

Theorem

$\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$ form a partition of the integers.

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Lemma

$\mathrm{Spec}(\alpha)$ has $N(\alpha, n) = \lceil (n+1)/\alpha \rceil - 1$ elements not larger than n.

Indeed:

$$
\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor k\alpha \rfloor \leqslant n] \quad \text{(recall that Spec}(\alpha)\text{ is a multiset)} \\
&= \sum_{k>0} [\lfloor k\alpha \rfloor < n+1] \\
&= \sum_{k>0} [k\alpha < n+1] \\
&= \sum_{k} [0 < k < (n+1)/\alpha] \\
&= \lceil (n+1)/\alpha \rceil - 1 \ \text{ by the formula in the “intervals” slide}
\end{aligned}
$$

Spectra and partitions of integers

Theorem

$\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$ form a partition of the integers.

Lemma

$\mathrm{Spec}(\alpha)$ has $N(\alpha,n) = \lceil (n+1)/\alpha \rceil - 1$ elements not larger than n.

As $\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$ have no repetitions, we only need to prove that $N(\sqrt{2},n) + N(2+\sqrt{2},n) = n$:

$$
\begin{aligned}
N(\sqrt{2},n) + N(2+\sqrt{2},n) &= \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1 \\
&= \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor \quad \text{because both are noninteger} \\
&= \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \\
&= (n+1) \underbrace{\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} \right)}_{=1} - \underbrace{\left(\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \right)}_{=1} \\
&= n+1-1 = n
\end{aligned}
$$

because if x and y are both noninteger but $x+y$ is integer, then $\{x\}+\{y\}=1$.

Next section

Floor/Ceiling Recurrences: Examples

The Knuth numbers:

$$K_0 = 1\,;$$
$$K_{n+1} = 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}) \text{ for } n \geqslant 0\,.$$

The sequence begins with:

$$K = \langle 1, 3, 3, 4, 7, 7, 7, 9, 9, 10, 13, \ldots \rangle$$

Merge sort $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$ records, number of comparisons:

$$f_1 = 0\,;$$
$$f_{n+1} = f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + n - 1 \text{ for } n > 1\,.$$

The sequence begins with:

$$f = \langle 0, 1, 3, 5, 8, 11, 14, 17, 21, 25, 29, 33 \ldots \rangle$$

Floor/Ceiling Recurrences: More Examples

The Josephus problem numbers:

$$J(1) = 1\,;$$
$$J(n) = 2J(\lfloor n/2 \rfloor) + (-1)^{n+1} \text{ for } n > 1\,.$$

The sequence begins as

$$J = \langle 1, 1, 3, 1, 3, 5, 7, 1, 3, 5, \ldots \rangle$$

Generalization of Josephus problem

Josephus problem in general: from n elements, every q-th is circularly eliminated. The element with number $J_q(n)$ will survive.

Theorem

$$J_q(n) = qn + 1 - D_k$$

where k is the smallest integer such that $D_k > (q-1)n$ and D_k is computed using the following recurrence relation:

$$D_0 = 1\,;$$

$$D_k = \left\lceil \frac{q}{q-1} D_{k-1} \right\rceil \text{ for every } k > 0\,.$$

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$$D_0 = 1\,;$$
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For example, if $q = 5$ and $n = 12$

$$D = \langle 1,2,3,4,5,7,9,12,15,19,24,30,38,48,60,75\ldots \rangle$$

Then $(q-1)n = 4 \cdot 12 = 48$, the proper D_k is $D_{14} = 60$, and

$$J_5(12) = 5 \cdot 12 + 1 - D_{14} = 60 + 1 - 60 = 1$$

Generalization of Josephus problem

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Sanity check: $q = 2$

Then $D_k = \left\lceil \frac{2}{2-1} D_{k-1} \right\rceil = 2D_{k-1}$ for every $k \geqslant 1$, so $D_k = 2^k$.
If $n = 2^m + \ell$, then $k = m+1$ and:

$$J_2(n) = 2 \cdot (2^m + \ell) + 1 - 2^{m+1} = 2\ell + 1$$

Proof of the Theorem

Whenever a person is passed over, we can assign a new number, as in the example below fo $n = 12, q = 5$

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16		17	18	19	20		21	22
23	24		25		26	27	28			29	30
31	32				33	34	35			36	
37	38				39	40				41	
42	43				44					45	
46	47				48						
49	50				51						
52					53						
54					55						
56											
57											
58											
59											
60											

Denoting by N and N' the current and previous element in a column, we get:

$$N = \left\lfloor \frac{N' - n - 1}{q - 1} \right\rfloor + N' - n$$

Proof of the Theorem (2)

Denoting by $D = qn+1-N$ and $D' = qn+1-N'$, we rewrite:

$$N = \left\lfloor \frac{N'-n-1}{q-1} \right\rfloor + N' - n$$

as:

$$qn+1-D = \left\lfloor \frac{qn+1-D'-n-1}{q-1} \right\rfloor + qn+1-D'-n$$

Let us transform this:

$$\begin{aligned}
D &= qn+1-\left\lfloor \frac{qn+1-D'-n-1}{q-1} \right\rfloor - qn-1+D'+n \\
&= D'+n-\left\lfloor \frac{n(q-1)-D'}{q-1} \right\rfloor = D'+n-\left\lfloor n-\frac{D'}{q-1} \right\rfloor \\
&= D'-\left\lfloor \frac{-D'}{q-1} \right\rfloor \\
&= D'+\left\lceil \frac{D'}{q-1} \right\rceil \\
&= \left\lceil \frac{q}{q-1} D' \right\rceil
\end{aligned}$$

Q.E.D.

Next section

‘mod’: The Binary Operation

If n and m are positive integers

Write $n = q \cdot m + r$ with $q, r \in \mathbb{N}$ and $0 \leqslant r < m$. Then:

$$q = \lfloor n/m \rfloor \text{ and } r = n - m \cdot \lfloor n/m \rfloor = n \bmod m$$

If x and y are real numbers

We follow the same idea and set:

$$x \bmod y = x - y \cdot \lfloor x/y \rfloor \quad \forall x, y \in \mathbb{R},\ y \neq 0$$

Note that, with this definition:

$$\begin{array}{lclcl}
5 \bmod 3 & = & 5 - 3 \cdot \lfloor 5/3 \rfloor & = & 5 - 3 \cdot 1 & = & 2 \\
5 \bmod -3 & = & 5 - (-3) \cdot \lfloor 5/(-3) \rfloor & = & 5 + 3 \cdot (-2) & = & -1 \\
-5 \bmod 3 & = & -5 - 3 \cdot \lfloor -5/3 \rfloor & = & -5 - 3 \cdot (-2) & = & 1 \\
-5 \bmod -3 & = & -5 - (-3) \cdot \lfloor -5/(-3) \rfloor & = & -5 + 3 \cdot 1 & = & -2
\end{array}$$

For $y = 0$ we want to respect the general rule that $x - (x \bmod y) \in y\mathbb{Z} = \{yk \mid k \in \mathbb{Z}\}$. This is done by:

$$x \bmod 0 = x$$

Properties of the $\mod$ operation

$x = \lfloor x \rfloor + x \bmod 1$

For $y = 1$ it is $x \bmod 1 = x - 1 \cdot \lfloor x/1 \rfloor = x - \lfloor x \rfloor$.

In other words: $x \bmod 1 = \{x\}$.

The distributive law: $c(x \bmod y) = cx \bmod cy$

If $c = 0$ both sides vanish; if $y = 0$ both sides equal cx. Otherwise:

$$c(x \bmod y) = c(x - y \lfloor x/y \rfloor) = cx - cy \lfloor cx/cy \rfloor = cx \bmod cy$$

TAL TECH

Warmup: Solve the following recurrence

$$
\begin{aligned}
X_n &= n && \text{for } 0 \leqslant n < m\,, \\
X_n &= X_{n-m} + 1 && \text{for } n \geqslant m\,.
\end{aligned}
$$

Warmup: Solve the following recurrence

$$X_n = n \quad \text{for } 0 \leqslant n < m\,,$$
$$X_n = X_{n-m} + 1 \quad \text{for } n \geqslant m\,.$$

Solution

We plot the first values when $m = 4$:

n	0	1	2	3	4	5	6	7	8	9
X_n	0	1	2	3	1	2	3	4	2	3

We conjecture that:

$$\text{if } n = qm + r \text{ with } q, r \in \mathbb{N} \text{ and } 0 \leqslant r < m \text{ then } X_n = q + r\ :$$

which clearly yields $X_n = \lfloor n/m \rfloor + n \bmod m$.

- Induction base: True for $n = 0, 1, \ldots, m-1$.
- Inductive step: Let $n \geqslant m$. If $X_{n'} = q' + r'$ for every $n' = q'm + r' < n = qm + r$, then:
$$X_n = X_{n-m} + 1 = X_{(q-1)m+r} + 1 = q - 1 + r + 1 = q + r$$

TAL TECH

Next section

Floor/Ceiling Sums

Example: Find a closed form for $\sum_{0\leqslant k<n}\left\lfloor\sqrt{k}\right\rfloor$

$$\begin{aligned}
\sum_{0\leqslant k<n}\left\lfloor\sqrt{k}\right\rfloor &= \sum_{k,m\geqslant 0} m\,[k<n]\left[m=\left\lfloor\sqrt{k}\right\rfloor\right] \\
&= \sum_{k,m\geqslant 0} m\,[k<n]\left[m\leqslant\sqrt{k}<m+1\right] \\
&= \sum_{k,m\geqslant 0} m\,[k<n]\left[m^2\leqslant k<(m+1)^2\right] \\
&= \underbrace{\sum_{k,m\geqslant 0} m\left[m^2\leqslant k<(m+1)^2\leqslant n\right]}_{=S_1} \\
&\quad + \underbrace{\sum_{k,m\geqslant 0} m\left[m^2\leqslant k<n<(m+1)^2\right]}_{=S_2}
\end{aligned}$$

Floor/Ceiling Sums

Example: Find a closed form for $\sum_{0\leqslant k<n} \left\lfloor \sqrt{k} \right\rfloor$

Case $n = a^2$, for a value $a \in \mathbb{N}$

Then $S_2 = 0$, while:

$$\begin{aligned}
S_1 &= \sum_{k,m\geqslant 0} m\left[m^2 \leqslant k < (m+1)^2 \leqslant a^2\right] \\
&= \sum_{m\geqslant 0} m((m+1)^2 - m^2)[m+1 \leqslant a] \\
&= \sum_{m\geqslant 0} m(2m+1)[m<a] \\
&= \sum_{m\geqslant 0} (2m(m-1)+3m)[m<a] \\
&= \sum_{m\geqslant 0} (2m^{\underline{2}}+3m^{\underline{1}})[m<a] = \sum_0^a (2m^{\underline{2}}+3m^{\underline{1}})\delta m \\
&= \left(\frac{2}{3}m^{\underline{3}} + \frac{3}{2}m^{\underline{2}}\right)\Big|_0^a = \frac{2}{3}a(a-1)(a-2) + \frac{3}{2}a(a-1) \\
&= \frac{2}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a = \frac{1}{6}a(a-1)(4a+1)
\end{aligned}$$

Floor/Ceiling Sums

Example: Find a closed form for $\sum_{0 \leqslant k < n} \lfloor \sqrt{k} \rfloor$

Case $n \neq b^2$, for any integer b

Let $a = \lfloor \sqrt{n} \rfloor$. Then:

- For $0 \leqslant k < a^2$ we get $S_1 = \frac{2}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a$ and $S_2 = 0$, as before.
- For $a^2 \leqslant k < n$, it is $S_1 = 0$ and:

$$
\begin{aligned}
S_2 &= \sum_{k,m \geqslant 0} m \left[m^2 \leqslant k < n < (m+1)^2 \right] \\
&= \sum_k a \left[a^2 \leqslant k < n \right] \\
&= a \sum_k \left[a^2 \leqslant k < n \right] \\
&= a(n - a^2) = an - a^3
\end{aligned}
$$

Floor/Ceiling Sums

Example: Find a closed form for $\sum_{0\leqslant k<n} \left\lfloor \sqrt{k} \right\rfloor$

To summarize:

$$\sum_{0\leqslant k<n} \left\lfloor \sqrt{k} \right\rfloor = \frac{2}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a + an - a^3$$

$$= an - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a \text{ where } a = \left\lfloor \sqrt{n} \right\rfloor$$