Binomial Coefficients and Generating Functions

ITT9132 Concrete Mathematics

TAL TECH

Contents

Next section

Binomial coefficients

Definition

Let r be a complex number and k an integer. The binomial coefficient "r choose k" is the complex number

$$\binom{r}{k} = \begin{cases} \dfrac{r \cdot (r-1) \cdots (r-k+1)}{k!} = \dfrac{r^{\underline{k}}}{k!} & \text{if } k \geqslant 0\,, \\ 0 & \text{if } k < 0\,. \end{cases}$$

If $r = n$ is a natural number

In this case,

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}$$

is the number of ways we can choose k elements from a set of n elements, in any order.

Consistently with this interpretation,

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leqslant k \leqslant n\,, \\ 0 & \text{if } k > n\,. \end{cases}$$

The binomial theorem

Theorem 1

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

for any integer $n \geqslant 0$.

Proof. Expanding $(a+b)^n = (a+b)(a+b)\cdots(a+b)$ yields the sum of the 2^n products of the form $e_1 e_2 \cdots e_n$, where each e_i is a or b. These terms are composed by selecting from each factor $(a+b)$ either a or b. For example, if we select a k times, then we must choose b $n-k$ times. So, we can rearrange the sum as

$$(a+b)^n = \sum_{k=0}^{n} C_k a^k b^{n-k},$$

where the coefficient C_k is the number of ways to select k elements (k factors $(a+b)$) from a set of n elements (from the product of n factors $(a+b)\cdot(a+b)\cdots(a+b)$).
That is why the coefficient C_k is called "(from) n choose k" and denoted by $\binom{n}{k}$. Q.E.D.

Binomial coefficients and combinations

Theorem 2

The number of subsets with k elements of a set with n elements is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof:

- To choose a sequence of k *different* elements, we have n choices for the first element, $n-1$ for the second, ..., $n-k+1$ for the kth.
- In other words, there are $n(n-1)\cdots(n-k+1) = n^{\underline{k}}$ sequences of k different elements.
- But two different sequences with the same elements identify the same subsets.
- And each such subsequence corresponds to one of the $k!$ ways of sorting k objects. In conclusion, we have $\frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$ subsets with k elements of a set with n elements. Q.E.D.

Some other notations used for the "n choose k" in literature: $C_k^n, C(n,k), {}_nC_k, {}^nC_k$.

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Some other notations used for the "n choose k" in literature:
$C^n_k, C(n,k), {}_nC_k, {}^nC_k$.

Properties of Binomial Coefficients

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$: A set of n elements has 2^n subsets.
2. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n=0]$: In a nonempty set, the number of subsets with odd cardinality is equal to the number of sets with even cardinality.

Proof:

- Take $a = b = 1$ in the binomial theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

- Take $a = -1$ and $b = 1$:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = (-1+1)^n = [n=0]$$

Q.E.D.

Another Property For $n \geqslant 0$ Integer

Symmetry of binomial coefficients

3 $\binom{n}{k} = \binom{n}{n-k}$ for every $n \geqslant 0$.

Proof. For $0 \leqslant k \leqslant n$ direct conclusion from Theorem 2:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k};$$

otherwise, both sides vanish. Q.E.D.

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Symmetry of binomial coefficients

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Proof. For $0 \leqslant k \leqslant n$ direct conclusion from Theorem 2:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k};$$

otherwise, both sides vanish. Q.E.D.

Only if n is nonnegative!

For $n = -1$ and $k \geqslant 0$,

$$\binom{-1}{k} = \frac{(-1)^{\underline{k}}}{k!} = (-1)^k \text{ but } \binom{-1}{-1-k} = 0;$$

while for $k < 0$,

$$\binom{-1}{k} = 0 \text{ but } \binom{-1}{-1-k} = (-1)^{|k|-1}.$$

Yet Another Property

Recurrence formula

4

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$

Yet Another Property

Recurrence formula

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$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$

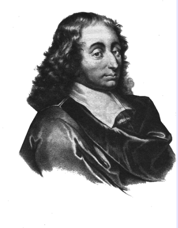
Proof: if $k \leqslant 0$ then both sides equal $[k=0]$; if $k > 0$, then:

$$\begin{aligned}
\binom{r-1}{k} + \binom{r-1}{k-1} &= \frac{(r-1)^{\underline{k}}}{k!} + \frac{(r-1)^{\underline{k-1}}}{(k-1)!} \\
&= \frac{(r-1)^{\underline{k-1}} \cdot (r-k)}{k!} + \frac{(r-1)^{\underline{k-1}} \cdot k}{k!} \\
&= \frac{(r-1)^{\underline{k-1}} \cdot (r-k+k)}{k!} \\
&= \frac{r \cdot (r-1)^{\underline{k-1}}}{k!} \\
&= \frac{r^{\underline{k}}}{k!} = \binom{r}{k},
\end{aligned}$$

Q.E.D.

Pascal's Triangle

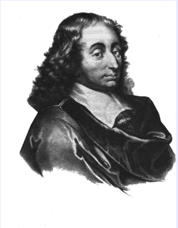
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1



Blaise Pascal
(1623–1662)

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

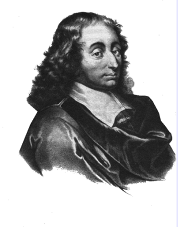


Blaise Pascal
(1623–1662)

- Pascal Triangle is symmetric with respect to the vertical line through its apex.
- Every number is the sum of the two numbers immediately above it.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1



Blaise Pascal
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Warmup: The hexagon property

Statement

For every $n \geqslant 2$ and $0 < k < n$,

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

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Interpretation

- Looking at Pascal's triangle in the previous slide, the six numbers in the expression above form a "hexagon" around $\binom{n}{k}$.
- Then the hexagon property says that the product of the odd-numbered corners of the hexagon equals that of the even-numbered corners.

Warmup: The hexagon property

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For every $n \geqslant 2$ and $0 < k < n$,

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

Proof

Consider the expression of the binomial coefficients as a ratio of products of primes.
At the numerator, both sides contribute with $(n-1)! \cdot n! \cdot (n+1)!$
At the denominator:

- The left hand side contributes with:

$$(k-1)! \cdot (n-k)! \cdot (k+1)! \cdot (n-k-1)! \cdot k! \cdot (n+1-k)!$$

- The right-hand side contributes with:

$$k! \cdot (n-1-k)! \cdot (k-1)! \cdot (n-k+1)! \cdot (k+1)! \cdot (n-k)!$$

The contributions of the two sides are thus equal, and the thesis follows.

The polynomial argument: Case study

Theorem

For every r complex and k integer,

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$

The polynomial argument: Case study

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Proof:

$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}$$

The polynomial argument: Case study

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For every r complex and k integer,

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$

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Wait! There's a problem:

We can have r appear in the lower index only if it is an integer!

The polynomial argument: Description

Lemma

If two polynomials with complex coefficients of degree at most k take the same values in more than k points, then they take the same values everywhere.

The polynomial argument: Description

Lemma

If two polynomials with complex coefficients of degree at most k take the same values in more than k points, then they take the same values everywhere.

Proof:

- Suppose $f(x)$ and $g(x)$ are polynomials of degree at most k taking the same values in the point $x_1, x_2, \ldots, x_{k+1}$.
- Then the polynomial $p(x) = f(x) - g(x)$ vanishes at each of the points $x_1, x_2, \ldots, x_{k+1}$, so there exists a polynomial $q(x)$ such that:

$$p(x) = (x - x_1)(x - x_2) \cdots (x - x_{k+1}) q(x) .$$

- But the left-hand side has degree at most k, while the right-hand side has degree at least $k+1$ unless $q(x) = 0$ everywhere.
- So $p(x) = 0$ everywhere too, that is, $f(x) = g(x)$ everywhere.

The polynomial argument: Description

Lemma

If two polynomials with complex coefficients of degree at most k take the same values in more than k points, then they take the same values everywhere.

So the proof in the previous slide was not too wrong ...

- $(r-k)\binom{r}{k}$ and $r\binom{r-1}{k}$ are polynomials in r of degree $k+1$.
- The two polynomials take the same value on every integer n.
- By the polynomial argument, they take the same value on every real number r.

Another application of the polynomial argument

Theorem

For every $r \in \mathbb{C}$ and $k, m \in \mathbb{Z}$:

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

Proof: If $k < 0$ or $m < k$ then both sides vanish. If $m \geqslant k \geqslant 0$, then for every $n \geqslant m$ integer:

$$\begin{aligned}
\binom{n}{m}\binom{m}{k} &= \frac{n!}{m!(n-m)!} \cdot \frac{m!}{k!(m-k)!} \\
&= \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{(m-k)!((n-k)-(m-k))!} \\
&= \binom{n}{k}\binom{n-k}{m-k},
\end{aligned}$$

whence the thesis by the polynomial argument.

Expanding the Addition Formula (1)

Summation on the upper index

For every $r \in \mathbb{C}$ and $n \in \mathbb{Z}$:

$$\sum_{k \leqslant n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

Expanding the Addition Formula (1)

Summation on the upper index

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Proof: if $n < 0$ then both sides are zero, otherwise we unfold:

$$\begin{aligned}
\binom{r+n+1}{n} &= \binom{r+n}{n} + \binom{r+n}{n-1} \\
&= \binom{r+n}{n} + \binom{r+n-1}{n-1} + \binom{r-n-1}{n-2} \\
&= \dots \\
&= \binom{r+n}{n} + \binom{r+n-1}{n-1} + \dots + \binom{r+1}{1} + \binom{r+1}{0} \\
&= \binom{r+n}{n} + \binom{r+n-1}{n-1} + \dots + \binom{r+1}{1} + \binom{r}{0}
\end{aligned}$$

Expanding the Addition Formula (2)

Summation on the upper index

For every $n, m \in \mathbb{N}$,

$$\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}$$

Expanding the Addition Formula (2)

Summation on the upper index

For every $n, m \in \mathbb{N}$,

$$\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}$$

Proof:

$$\begin{aligned}
\binom{n+1}{m+1} &= \binom{n}{m+1} + \binom{n}{m} \\
&= \binom{n-1}{m+1} + \binom{n-1}{m} + \binom{n}{m} \\
&= \dots \\
&= \binom{0}{m+1} + \binom{0}{m} + \binom{1}{m} + \dots + \binom{n-1}{m} + \binom{n}{m} \\
&= \binom{0}{m} + \binom{1}{m} + \dots + \binom{n}{m}
\end{aligned}$$

because $m+1 \geqslant 1$, so surely $\binom{0}{m+1} = 0$.

Generalizing the Binomial Theorem

Generalized Binomial Theorem

Let either $r \in \mathbb{N}$ or $|x/y| < 1$. Then:

$$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}$$

For if r is arbitrary but $z = x/y$ is such that $|z| < 1$, then the Taylor series of $(1+z)^r$ in a neighborhood of the origin converges absolutely:

$$\begin{aligned}(1+z)^r &= 1 + r(1+0)^{r-1}z + \frac{r^{\underline{2}}(1+0)^{r-2}}{2!}z^2 + \frac{r^{\underline{3}}(1+0)^{r-3}}{3!}z^3 + \frac{r^{\underline{4}}(1+0)^{r-4}}{4!}z^4 + \cdots \\ &= \sum_k \binom{r}{k} z^k = \sum_k \binom{r}{k} \left(\frac{x}{y}\right)^k,\end{aligned}$$

and multiplying both sides by y^r we get the thesis.

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and multiplying both sides by y^r we get the thesis.

This is actually our first use of a generating function.

Warmup: The binomial inversion formula

Theorem (Binomial inversion formula)

Let f and g be two complex- functions defined on $\mathbb{N}$. The following are equivalent:

1. For every $n \geqslant 0$, $g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k)$.
2. For every $n \geqslant 0$, $f(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(k)$.

Note that the role of f and g is symmetrical.

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Note that the role of f and g is symmetrical.

Proof

If $g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k)$ for every $n \geqslant 0$, then for every $n \geqslant 0$ also:

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-1)^k g(k) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{m=0}^{k} \binom{k}{m} (-1)^m f(m) \\
&= \\
&= \sum_{0 \leqslant m \leqslant k \leqslant m} \frac{n^{\underline{m}} (n-m)^{\underline{k-m}}}{k!} \frac{k!}{m!(k-m)!} (-1)^{k-m} f(m) \\
&= \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{k=m}^{n} (-1)^{k-m} \binom{n-m}{k-m} \right) f(m) \\
&= \sum_{m=0}^{n} \binom{n}{m} [m=n] \, f(m) = f(n).
\end{aligned}
$$

Binomial identities cheat sheet

- $\binom{r}{0} = 1$ for every r complex.
- $\binom{r}{1} = r$ for every r complex.
- $\binom{n}{n} = [n \geqslant 0]$ for every n integer.
- $\binom{0}{k} = [k = 0]$ for every k integer.
- $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$ for every r complex and $k \neq 0$ integer.
- $k\binom{r}{k} = r\binom{r-1}{k-1}$ for every r complex and k integer (also 0).
- $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ for every r complex and k integer.
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for every $n \geqslant 0$ integer.
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n = 0]$ for every $n \geqslant 0$ integer.
- $\sum_{k \leqslant n} \binom{r+k}{k} = \binom{r+n+1}{n}$ for every r complex and n integer.
- $\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}$ for every $n, m \in \mathbb{N}$.

Next section

A Sum of Ratios

Problem

Compute $\sum_{k=0}^{m} \binom{m}{k} / \binom{n}{k}$, where $n \geqslant m \geqslant 0$ are integers.

A Sum of Ratios

Problem

Compute $\sum_{k=0}^{m} \binom{m}{k} / \binom{n}{k}$, where $n \geqslant m \geqslant 0$ are integers.

Let us work on the summand:

$$\frac{\binom{m}{k}}{\binom{n}{k}} = \frac{\binom{n}{m}\binom{m}{k}}{\binom{n}{k}\binom{n}{m}} = \frac{\binom{n}{k}\binom{n-k}{m-k}}{\binom{n}{k}\binom{n}{m}} = \frac{\binom{n-k}{m-k}}{\binom{n}{m}}$$

which depends on k in the numerator, but not the denominator. Now:

$$\sum_{k=0}^{n} \binom{n-k}{m-k} = \sum_{k \geqslant 0} \binom{n-k}{m-k} = \sum_{k \leqslant m} \binom{n-m+k}{k} = \binom{(n-m)+m+1}{m} = \binom{n+1}{m}$$

by renaming k as $m-k$. In conclusion:

$$\sum_{k=0}^{m} \binom{m}{k} \Big/ \binom{n}{k} = \frac{\binom{n+1}{m}}{\binom{n}{m}} = \frac{(n+1)!}{m!(n+1-m)!} \cdot \frac{m!(n-m)!}{n!} = \frac{n+1}{n+1-m} \, .$$

Next section

Power series of functions

Example: Functions expanded as power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Power series of functions (2)

Power series of a function

- The *power series expansion* of the function f in a neighborhood of a point c is an expression of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots,$$

  where $c, a_0, a_1, \ldots$ are constants. (Taylor series)
- The special case $c = 0$ provide the Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

- The coefficients are defined as

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Power series of functions (3)

Example: Generating functions

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \qquad \langle 1, 1, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{24}, \cdots \rangle$$

$$\sin(x) = x - \frac{x^3}{3} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \qquad \langle 0, 1, -\tfrac{1}{3}, 0, \tfrac{1}{120}, 0, -\tfrac{1}{5040}, \cdots \rangle$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \langle 1, 1, 1, 1, \cdots \rangle$$

Generating functions ... ... of the sequences

Generating Functions

Definition

The generating function of the sequence $\langle g_0, g_1, g_2, \ldots \rangle$ is the power series

$$G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots = \sum_{n=0}^{\infty} g_n x^n$$

Some simple examples

$$\begin{array}{ccc}
\langle 0,0,0,0,\ldots \rangle & \longleftrightarrow & 0 + 0x + 0x^2 + 0x^3 + \cdots = 0 \\
\langle 1,0,0,0,\ldots \rangle & \longleftrightarrow & 1 + 0x + 0x^2 + 0x^3 + \cdots = 1 \\
\langle 2,3,1,0,\ldots \rangle & \longleftrightarrow & 2 + 3x + 1x^2 + 0x^3 + \cdots = 2 + 3x + 1x^2
\end{array}$$

More examples (1)

$\langle 1,1,1,1,\ldots \rangle \quad \longleftrightarrow \quad 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots \\ xS &= \phantom{1 + {}} x + x^2 + x^3 + \cdots \end{aligned}$$

Subtracting:

$$(1-x)S = 1\,, \text{ that is, } S = \frac{1}{1-x}$$

More examples (1)

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Subtracting:

$$(1-x)S = 1\,, \text{ that is, } S = \frac{1}{1-x}$$

NB! This formula converges only for $-1 < x < 1$.

We will see that we don't need to worry about convergence issues.

More examples (2)

$\langle a, ab, ab^2, ab^3, \dots \rangle \longleftrightarrow a + abx + ab^2x^2 + ab^3x^3 + \cdots = \frac{a}{1-bx}$

Like in the previous example:

$$\begin{aligned} S &= a + abx + ab^2x^2 + ab^3x^3 + \cdots \\ bxS &= \phantom{a + {}} abx + ab^2x^2 + ab^3x^3 + \cdots \end{aligned}$$

Subtract and get:

$$(1 - bx)S = a\,, \text{ that is, } S = \frac{a}{1 - bx}$$

More examples (3)

Taking in the last example $a = 0{,}5$ and $b = 1$ yields

$$0{,}5 + 0{,}5x + 0{,}5x^2 + 0{,}5x^3 + \cdots = \frac{0{,}5}{1-x} \tag{1}$$

Taking $a = 0{,}5$ and $b = -1$, gives

$$0{,}5 - 0{,}5x + 0{,}5x^2 - 0{,}5x^3 + \cdots = \frac{0{,}5}{1+x} \tag{2}$$

Adding equations (1) and (2), we get the generating function of the sequence $\langle 1, 0, 1, 0, 1, 0, \ldots \rangle$:

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{0{,}5}{1-x} + \frac{0{,}5}{1+x} = \frac{1}{(1-x)(1+x)} = \frac{1}{1-x^2}$$

Next subsection

The complex derivative

Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, and z an internal point of A.
The complex derivative of f in z is (if exists) the quantity

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Complex differentiation follows the same rules as real differentiation:

- $(af(z) + bg(z))' = af'(z) + bg'(z)$.
- $(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$.
- $(f(g(z)))' = f'(g(z))g'(z)$.
- If $g(z) = f^{-1}(z)$, then $g'(z) = 1/f'(g(z))$.

If A is open and f has complex derivative in every point of A, e say that f is holomophic in A.

The complex derivative and the partial derivatives

Let $f(z) = u(z) + iv(z)$. If $f'(z)$ exists, then:

- For $\Delta z = \Delta x$,

$$\frac{\partial f}{\partial x}(z) = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = f'(z)$$

- For $\Delta z = i\Delta y$,

$$\frac{\partial f}{\partial y}(z) = i \lim_{\Delta x \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} = if'(z)$$

As $i \cdot i = -1$, we get the Cauchy-Riemann conditions:

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0\,, \text{ that is, } u_x = v_y \text{ and } u_y = -v_x\,.$$

Vice versa, if f has continuous partial derivatives and satisfies the Cauchy-Riemann conditions, then f has a complex derivative.

Elementary holomorphic functions

$f(z)$	holomorphic in	defined as	$f'(z)$
$z^m,\ m \in \mathbb{N}$	$z \in \mathbb{C}$	$\prod_{i=1}^{m} z$	mz^{m-1}
$z^m,\ m \in \mathbb{Z},\ m < 0$	$z \in \mathbb{C} \setminus \{0\}$	$\prod_{i=1}^{m} z^{-1}$	mz^{m-1}
e^z	$z \in \mathbb{C}$	$e^x(\cos y + i \sin y)$ where $z = x + iy$	e^z
$\sin z$	$z \in \mathbb{C}$	$\dfrac{e^{iz} - e^{-iz}}{2i}$	$\cos z$
$\cos z$	$z \in \mathbb{C}$	$\dfrac{e^{iz} + e^{-iz}}{2}$	$-\sin z$
$\log z$	$z \in \mathbb{C} \setminus (-\infty, 0]$	$\ln \lvert z \rvert + i \arg z$	$\dfrac{1}{z}$

where $\arg z$ is the unique $\theta \in (-\pi, \pi]$ such that $z = |z| \cdot (\cos\theta + i \sin\theta)$.

Warmup: The complex exponential

Idea

We look for a solution $f : \mathbb{C} \to \mathbb{C}$ to the Cauchy problem:

$$\begin{cases} f'(z) & = & f(z) \ \forall z \in \mathbb{C}, \\ f(0) & = & 1\,. \end{cases}$$

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- As $f' = \partial f / \partial x$ when the left-hand side exists, we can search:

$$f(z) = e^x \cdot g(y) \ \text{where} \ x + iy = z\,.$$

- The Cauchy-Riemann conditions and the hypothesis tell us that:

$$e^x g'(y) = if'(z) = if(z)\,.$$

- Consequently, $e^x g''(y) = -f(z)$, and by dividing both sides by e^x,

$$g''(y) = -g(y) \ \forall y \in \mathbb{R}\,.$$

- Then it must be $g(y) = A \cos y + B \sin y$ for suitable $A, B \in \mathbb{C}$.

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$$\begin{cases} f'(z) & = & f(z) \ \forall z \in \mathbb{C}, \\ f(0) & = & 1\,. \end{cases}$$

- Summarizing:

$$f(z) = e^x (A \cos y + B \sin y) \text{ where } z = x + iy\,.$$

- $f(0) = 1$ yields $A = 1$; $f'(0) = 1$ yields $B = i$.

We conclude:

$$e^z = e^x (\cos y + i \sin y) \text{ where } z = x + iy\,.$$

Warmup: The complex exponential

Idea

We look for a solution $f : \mathbb{C} \to \mathbb{C}$ to the Cauchy problem:

$$\begin{cases} f'(z) &= f(z) \ \forall z \in \mathbb{C}, \\ f(0) &= 1 . \end{cases}$$

Remark

Similarly, the complex cosine and sine are the solutions of the Cauchy problems:

$$\begin{cases} g''(z) &= -v(z) \ \forall z \in \mathbb{C}, \\ g(0) &= 1 , \\ g'(0) &= 0 ; \end{cases} \qquad \begin{cases} h''(z) &= -w(z) \ \forall z \in \mathbb{C}, \\ h(0) &= 0 , \\ h'(0) &= 1 . \end{cases}$$

As e^{iz} and e^{-iz} both satisfy $f'' = -f$ in $\mathbb{C}$, setting the initial conditions yields:

$$g(z) = \cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } h(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i} .$$

Warmup: The complex exponential

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We look for a solution $f : \mathbb{C} \to \mathbb{C}$ to the Cauchy problem:

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As e^{iz} and e^{-iz} both satisfy $f'' = -f$ in $\mathbb{C}$, setting the initial conditions yields:

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We then recover Euler's equation:

$$e^{iz} = \cos z + i \sin z \ \text{ for every } z \in \mathbb{C}\,.$$

Convergence of sequences of functions

Pointwise convergence

Let $f_n : A \to \mathbb{C}$ be functions. The (pointwise) limit of the sequence $\{f_n\}_{n\geqslant 0}$ is the function defined by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for every $z \in A$ where the limit exists.

For power series: $\sum_{n\geqslant 0} a_n z^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n$.

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For power series: $\sum_{n \geqslant 0} a_n z^n = \lim_{N \to \infty} \sum_{n=0}^{N} a_n z^n$.

Uniform convergence

The sequence of functions $\{f_n\}_{n \geqslant 0}$ of functions converges uniformly to f in A if:

$$\forall \varepsilon > 0 \, \exists n_\varepsilon \geqslant 0 \text{ such that } \forall n > n_\varepsilon \, \forall z \in A \,.\, |f_n(z) - f(z)| < \varepsilon :$$

that is, if pointwise convergence is independent of the point.

- The sequence $f_n(x) = e^{-x^2} [|x| \leqslant n]$ converges to $f(x) = e^{-x^2}$ uniformly in $\mathbb{R}$.
- The sequence $f_n(x) = [x > n]$ converges to zero in $\mathbb{R}$, but not uniformly.

Consequences of uniform convergence

Continuity of the limit

Uniform limit of continuous functions is continuous.

Not true for simply pointwise convergence:

$$\text{if } f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leqslant x \leqslant 1/n\,, \\ 0 & \text{if } 1/n \leqslant x \leqslant 1\,, \end{cases} \quad \text{then } \lim_{n\to\infty} f_n(x) = [x = 0]\,.$$

Swap limits

If f_n converges uniformly in A, then:

$$\lim_{z\to z_0} \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \lim_{z\to z_0} f_n(z)\ \forall z_0 \in A$$

Swap limit with differentiation

If $f_n \to f$ uniformly in A, all the f_n are differentiable, and f_n' converges uniformly, then f is differentiable and:

$$f'(z) = \lim_{n\to\infty} f_n'(z)$$

Total convergence

Definition

A series of functions $\sum_k f_k(z)$ converges totally in a set A if there exists a convergent series $\sum_k b_k$ with nonnegative coefficients such that

$$|f_k(z)| \leqslant b_k \text{ for every } z \in A$$

Total convergence is the strongest form of convergence for series of functions:

$$\begin{tikzcd}
& \text{total} \arrow[d] & \\
& \text{uniform absolute} \arrow[dl] \arrow[dr] & \\
\text{absolute} \arrow[dr] & & \text{uniform} \arrow[dl] \\
& \text{pointwise} &
\end{tikzcd}$$

The convergence radius of a power series

Definition

The convergence radius of the power series

$$S(z) = \sum_{n \geqslant 0} a_n (z-c)^n$$

is:

$$R = \frac{1}{\limsup_{n \geqslant 0} \sqrt[n]{|a_n|}},$$

with the conventions $1/0 = \infty$, $1/\infty = 0$.

Examples

- For $\alpha \in \mathbb{C}$, $\sum_{n \geqslant 0} \alpha^n z^n$ has convergence radius $1/|\alpha|$.
- $\sum_{n \geqslant 1} \dfrac{z^n}{n}$ has convergence radius 1.
- $\sum_{n \geqslant 0} \dfrac{z^n}{n!}$ has infinite convergence radius.

The Abel-Hadamard theorem

Statement

Let $S(z)$ be a power series of center c and convergence radius R.

1. If $R > 0$, then $S(z)$ converges totally on every closed and bounded subset of the open disk of center c and radius R.
2. If $R < \infty$, then $S(z)$ does not converge at any point z such that $|z - c| > R$.

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Examples

- $\sum_{n\geqslant 0} \dfrac{(-1)^n}{2^n(n+1)} z^n$ converges totally in $\{|z|\leqslant 1\}$.
- $\sum_{n\geqslant 0} \dfrac{(2i)^n}{n+1}$ does not exist.

If $|z-z_0|=R$ then "all bets are equal":

- $\sum_{n\geqslant 1} \dfrac{z^n}{n}$ has $R=1$ and does not converge at $z=1$.
- $\sum_{n\geqslant 1} \dfrac{z^n}{n^2}$ has $R=1$ and converges totally on the closed unit disk.

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Consequence for generating functions

If $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} < \infty$,
then the generating function of g_n is well defined in a neighborhood of 0.

Exploiting power series

Let $S(z) = \sum_{n \geqslant 0} a_n (z-c)^n$ for $|z-c| < r$.

1. For any such z we can approximate $S(z)$ with its partial sum

$$S_N(z) = \sum_{0 \leqslant n \leqslant N} a_n (z-c)^n$$

2. The quantity $|S(z) - S_N(z)|$ can be made arbitrarily small by setting N large enough.
3. The choice of n can be made good for every z such that $|z-c| \leqslant \rho < r$.
4. Arithmetic operations are sufficient to compute $S_N(z)$.

Power series are holomorphic functions

- Let $S(z) = \sum_{n \geqslant 0} a_n (z-c)^n$ and let $R > 0$ be its convergence radius.
- The function

$$T(z) = \sum_{n \geqslant 0} \frac{d}{dz} (a_n (z-c)^n) = \sum_{n \geqslant 1} n a_n (z-c)^{n-1} = \sum_{n \geqslant 0} (n+1) a_{n+1} (z-c)^n$$

  is still a power series.
- But

$$\limsup_{n \geqslant 0} \sqrt[n]{|(n+1)a_{n+1}|} = \limsup_{n \geqslant 0} \sqrt[n]{|a_n|} :$$

  so $T(z)$ also has convergence radius R.
- By the Abel-Hadamard theorem, for $|z-c| < R$,

$$S'(z) = \sum_{n \geqslant 0} (n+1) a_{n+1} (z-c)^n = T(z)$$

  Note how we swapped differentiation with sum, which is made possible by uniform convergence, which in turn is ensured by total convergence.

TAL TECH

Holomorphic functions are power series *locally*

Laurent's theorem

Let f be holomorphic in a disk

$$D_r(c) = \{z \in \mathbb{C} \mid |z - c| < r\}.$$

Then there exist a sequence $\{a_n\}_{n \geqslant 0}$ of complex numbers such that:

1. The power series $S(z) = \sum_{n \geqslant 0} a_n (z - c)^n$ has convergence radius $R \geqslant r$.
2. For every $z \in D_r(c)$ we have $S(z) = f(z)$.

A function which is "locally a power series" at each point is called analytic.
For complex functions of a complex variable, analyticity is the same as holomorphy.

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Counterexample in real analysis

Let $f(x) = e^{-1/x^2}$ for $x \neq 0$, $f(0) = 0$.

- Then f is infinitely differentiable in $\mathbb{R}$ and nonzero everywhere except $x = 0$ ...
- ... but the Taylor series in $x = 0$ vanishes!

Holomorphic functions are power series *locally*

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For complex functions of a complex variable, analyticity is the same as holomorphy.

Consequence for generating functions

Every function that is analytic in a neighborhood of the origin
is the generating function of some sequence.

The identity principle for analytic functions

Statement

- Let A be a connected open subset of the complex plane.
- Let $f : A \to \mathbb{C}$ be an analytic function.
- Suppose f is not identically zero in A.
- Then all the zeroes of f in A are isolated:
  If $z_0 \in A$ and $f(z_0) = 0$, then there exists $r > 0$ such that $f(z) \neq 0$ for every z such that $0 < |z - z_0| < r$.

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Corollary: Uniqueness of analytic continuation

Let:

- I a nonempty interval of the real line;
- A a connected open subset of the complex plane such that $I \subseteq A$; and
- $f : I \to \mathbb{R}$ a continuous function.

Then there exists at most one function analytic in A which coincides with f on I.

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  If $z_0 \in A$ and $f(z_0) = 0$, then there exists $r > 0$ such that $f(z) \neq 0$ for every z such that $0 < |z - z_0| < r$.

Consequence for generating functions

Every sequence $\{g_n\}_{n \geqslant 0}$ of complex numbers such that $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} < \infty$ is uniquely determined by its generating function.

$1+2+3+4+\ldots=-1/12$!??

The series

$$\sum_{n\geqslant 1} n^{-s}$$

converges for every real value $s>1$: for example, for $s=2$,

$$\sum_{n\geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The Riemann zeta function is the unique analytic function $\zeta(s)$, defined for $s\in\mathbb{C}\setminus\{1\}$, such that $\zeta(s)=\sum_{n\geqslant 1} n^{-s}$ for every real $s>1$.

- It happens that $\zeta(-1)=-1/12$.
- This does not mean that $\sum_{n\geqslant 1} n=-1/12$!
- Instead, it means that the formula $\zeta(s)=\sum_{n\geqslant 1} n^{-s}$ can be assumed valid only when s is real and greater than 1.

Basic generating functions

$G(z)$	z	$\langle g_0, g_1, g_2, g_3, \ldots \rangle$	g_n
$z^m,\ m \in \mathbb{N}$	$z \in \mathbb{C}$	$\langle 0, \ldots, 0, 1, 0, \ldots, \rangle$	$[n = m]$
e^z	$z \in \mathbb{C}$	$\langle 1, 1, \frac{1}{2}, \frac{1}{6}, \ldots \rangle$	$\dfrac{1}{n!}$
$\cos z$	$z \in \mathbb{C}$	$\langle 1, 0, -\frac{1}{2}, 0, \ldots \rangle$	$\dfrac{(-1)^{\lfloor n/2 \rfloor}}{n!} \cdot [n \text{ is even}]$
$\sin z$	$z \in \mathbb{C}$	$\langle 0, 1, 0, -\frac{1}{6}, \ldots \rangle$	$\dfrac{(-1)^{\lfloor n/2 \rfloor}}{n!} \cdot [n \text{ is odd}]$
$(1+z)^\alpha$	$\lvert z \rvert < 1$	$\langle 1, \alpha, \frac{\alpha(\alpha-1)}{2}, \frac{\alpha^3}{6}, \ldots \rangle$	$\binom{\alpha}{n} = \dfrac{\alpha^{\underline{n}}}{n!}$
$\dfrac{1}{1-\alpha z}$	$\lvert z \rvert < 1/\lvert \alpha \rvert$	$\langle 1, \alpha, \alpha^2, \alpha^3, \ldots \rangle$	α^n
$\ln \dfrac{1}{1-z}$	$\lvert z \rvert < 1$	$\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$	$\dfrac{1}{n} \cdot [n > 0]$
$\ln(1+z)$	$\lvert z \rvert < 1$	$\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, \ldots \rangle$	$\dfrac{(-1)^{n-1}}{n} \cdot [n > 0]$

Recall our convention that, if a is infinite or undefined, then $a \cdot [\text{False}] = 0$.

Analytic functions and generating functions: A summary

1. Every function that is analytic in a neighborhood of the origin of the complex plane is the generating function of some sequence.
   Reason why: Laurent's theorem.
2. Every sequence $\{g_n\}_{n \geqslant 0}$ of complex numbers such that

$$\limsup_n \sqrt[n]{|g_n|} < \infty$$

   admits a generating function.
   Reason why: The Abel-Hadamard theorem.
3. Every such sequence is uniquely determined by its generating function.
   Reason why: Uniqueness of analytic continuation.

We can thus use all the standard operations on sequences and their generating functions, without caring about definition, convergence, etc., provided we do so under the tacit assumption that we are in a "small enough" circle centered in the origin of the complex plane.

Power series and infinite sums

The problem

- Consider an infinite sum of the form $\sum_{n \geqslant 0} a_n \beta^n$.
- Suppose that we are given a closed form for the generating function $G(z)$ of the sequence $\langle a_0, a_1, a_2, \ldots \rangle$.
- Can we deduce that $\sum_{n \geqslant 0} a_n \beta^n = G(\beta)$?

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- Can we deduce that $\sum_{n\geqslant 0} a_n\beta^n = G(\beta)$?

Answer: It depends!

Let R be the convergence radius of the power series $\sum_{n\geqslant 0} a_n z^n$.

- If $|\beta| < R$: YES by Abel-Hadamard and uniqueness of analytic continuation.
- If $|\beta| > R$: NO by Abel-Hadamard.
- If $|\beta| = R$: Sometimes yes, sometimes not!

Warmup: A fast approximation for $\ln 2$

Problem

Show that $\sum_{k\geqslant 1} \frac{1}{k\cdot 2^k} = \ln 2$.

Warmup: A fast approximation for $\ln 2$

Problem

Show that $\sum_{k \geqslant 1} \frac{1}{k \cdot 2^k} = \ln 2$.

Solution:

- The sum $\sum_{k \geqslant 1} \frac{1}{k \cdot 2^k}$ resembles the power series $\sum_{k \geqslant 1} \frac{z^k}{k}$ evaluated at $z = 1/2$.
- Since $\lim_{k \to \infty} \sqrt[k]{k} = 1$, $z = 1/2$ is within the convergence radius of the series.
- But for $|z| < 1$ it is $\sum_{k \geqslant 1} \frac{z^k}{k} = \ln \frac{1}{1-z}$: we can then evaluate indifferently the left-hand side or the right-hand side!
- As $\frac{1}{1-1/2} = 2$, the thesis follows.

Two rules for $|\beta| = R$

Abel's summation formula

Let $S(z) = \sum_{n \geqslant 0} a_n z^n$ be a power series with center 0 and convergence radius 1. If

$$S = \sum_{n \geqslant 0} a_n = S(1)$$

exists, then $S(z)$ converges uniformly in $[0,1]$. In particular,

$$S = \lim_{x \to 1^-} S(x) \ , \ x \in [0,1]$$

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The converse does not hold!

For $|z| < 1$ we have:

$$\sum_{n\geqslant 0} (-1)^n z^n = \frac{1}{1+z}$$

Then $L = \frac{1}{2}$ but S does not exist.

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Tauber's theorem

Let $S(z) = \sum_{n \geqslant 0} a_n z^n$ be a power series with center 0 and convergence radius 1. If

$$L = \lim_{x \to 1^-} S(x) \ , \ x \in [0,1]$$

exists and in addition $\lim_{n \to \infty} n a_n = 0$, then $S = S(1)$ also exists, and coincides with L.