Binomial Coefficients and Generating Functions

ITT9132 Concrete Mathematics

TAL TECH

Contents

Next section

Next subsection

Operations on Generating Functions

1. Linear combination

If:

$$\begin{array}{rcl} \langle f_0, f_1, f_2, \ldots \rangle & \longleftrightarrow & F(z) \\ \text{and } \langle g_0, g_1, g_2, \ldots \rangle & \longleftrightarrow & G(z) \end{array}$$

then for every $\alpha, \beta \in \mathbb{C}$:

$$\langle \alpha f_0 + \beta g_0, \alpha f_1 + \beta g_1, \alpha f_2 + \beta g_2, \ldots \rangle \quad \longleftrightarrow \quad \alpha F(z) + \beta G(z).$$

Proof:

$$\begin{array}{rcl} \langle \alpha f_0 + \beta g_0, \alpha f_1 + \beta g_1, \alpha f_2 + \beta g_2, \ldots \rangle & \longleftrightarrow & \sum_{n \geqslant 0} (\alpha f_n + \beta g_n) z^n \\ & = & \alpha \sum_{n \geqslant 0} f_n z^n + \beta \sum_{n \geqslant 0} g_n z^n \\ & & \text{by absolute convergence} \\ & = & \alpha F(z) + \beta G(z). \end{array}$$

Q.E.D.

Operations on Generating Functions (2)

2. Right-shift

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, then

$$\Big\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeros}}, g_0, g_1, g_2, \ldots \Big\rangle \longleftrightarrow z^k \cdot G(z).$$

Proof:

$$\begin{aligned}
\langle 0, 0, \ldots, 0, g_0, g_1, g_2, \ldots \rangle \quad &\longleftrightarrow \quad \sum_{n \geqslant k} g_{n-k} z^n \\
&= \quad z^k \sum_{n \geqslant 0} g_n z^n \text{ by absolute convergence} \\
&= \quad z^k \cdot G(z)
\end{aligned}$$

Q.E.D.

Operations on Generating Functions (3)

3. Left shift

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow F(z)$, then for every $n \geqslant 0$:

$$\langle g_k, g_{k+1}, g_{k+2}, \ldots \rangle \quad \longleftrightarrow \quad \frac{1}{z^k} \left(G(z) - \sum_{n=0}^{k-1} g_n z^n \right)$$

Proof:

$$\begin{aligned} \left\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeros}}, g_k, g_{k+1}, g_{k+2}, \ldots \right\rangle \quad & \longleftrightarrow \quad G(z) - \sum_{n=0}^{k-1} g_n z^n \\ & = \quad z^k \cdot \frac{1}{z^k} \left(G(z) - \sum_{n=0}^{k-1} g_n z^n \right), \end{aligned}$$

that is: $G(z) - \sum_{n=0}^{k-1} g_n z^n$ is the shift to the right by k positions of $\sum_{n \geqslant 0} g_{n+k} z^n$.
Q.E.D.

Operations on Generating Functions (4)

4. Differentiation

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, then:

$$\langle g_1, 2g_2, 3g_3, \ldots \rangle \quad \longleftrightarrow \quad G'(z).$$

Proof:

$$\begin{aligned}
\langle g_1, 2g_2, 3g_3, \ldots \rangle \quad &\longleftrightarrow \quad \sum_{n \geqslant 0} (n+1) g_{n+1} z^n \\
&= \quad \sum_{n \geqslant 1} g_n \, n \, z^{n-1} \\
&= \quad \sum_{n \geqslant 0} g_n \frac{d}{dz} z^n \\
&= \quad \frac{d}{dz} \sum_{n \geqslant 0} g_n z^n \quad \text{by uniform convergence} \\
&= \quad G'(z)
\end{aligned}$$

Q.E.D.

Operations on Generating Functions (4)

4. Differentiation

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, then:

$$\langle g_1, 2g_2, 3g_3, \ldots \rangle \quad \longleftrightarrow \quad G'(z).$$

Example

- $\langle 1,1,1,1,\ldots \rangle \longleftrightarrow \dfrac{1}{1-z}$
- $\langle 1,2,3,4,\ldots \rangle \longleftrightarrow \dfrac{1}{(1-z)^2}$
- $\langle 0,1,2,3,\ldots \rangle \longleftrightarrow \dfrac{z}{(1-z)^2}$

Operations on Generating Functions (4)

4. Differentiation

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, then:

$$\langle g_1, 2g_2, 3g_3, \ldots \rangle \quad \longleftrightarrow \quad G'(z).$$

Corollary

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, then:

$$\langle 0, g_1, 2g_2, 3g_3, \ldots \rangle \quad \longleftrightarrow \quad zG'(z).$$

Operations on Generating Functions (5)

5. Integration

If $\langle g_0, g_1, g_2, \dots \rangle \longleftrightarrow G(z)$, then:

$$\left\langle 0, g_0, \frac{g_1}{2}, \frac{g_2}{3}, \dots \right\rangle \quad \longleftrightarrow \quad \int_0^z G(w)\,dw = \int_0^1 zG(zt)\,dt\,.$$

Proof:

$$\begin{aligned}
\left\langle 0, f_0, \frac{1}{2}f_1, \frac{1}{3}f_2, \frac{1}{4}f_3, \dots \right\rangle \quad \longleftrightarrow \quad & f_0 z + \frac{1}{2}f_1 z^2 + \frac{1}{3}f_2 z^3 + \frac{1}{4}f_3 z^4 + \dots \\
= \quad & f_0 \int_0^z dw + f_1 \int_0^z w\,dw + f_2 \int_0^z w^2\,dw + f_3 \int_0^z w^3\,dw + \dots \\
= \quad & \int_0^z \left(f_0 + f_1 w + f_2 w^2 + f_3 w^3 + \dots\right) dw \\
& \text{by uniform convergence} \\
= \quad & \int_0^z F(w)\,dw
\end{aligned}$$

Q.E.D.

Operations on Generating Functions (5)

5. Integration

If $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, then:

$$\left\langle 0, g_0, \frac{g_1}{2}, \frac{g_2}{3}, \ldots \right\rangle \quad \longleftrightarrow \quad \int_0^z G(w)\,dw = \int_0^1 zG(zt)\,dt\,.$$

Example

- $\langle 1, 1, 1, 1, \ldots \rangle \longleftrightarrow \dfrac{1}{1-z}$
- $\left\langle 0, 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots \right\rangle \longleftrightarrow \int_0^z \frac{dw}{1-w} = \log \dfrac{1}{1-z}$

Operations on Generating Functions (6)

6. Convolution (product)

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(z)$, $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, and

$$h_n = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0 = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=k} a_i b_j$$

then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Operations on Generating Functions (6)

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then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Proof:

$$\begin{aligned} F(z) \cdot G(z) &= (f_0 + f_1 z + f_2 z^2 + \ldots) \cdot (g_0 + g_1 z + g_2 z^2 + \ldots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0) z + (f_0 g_2 + f_1 g_1 + f_2 g_0) z^2 + \ldots \\ &\quad \text{by absolute convergence} \end{aligned}$$

Q.E.D.

Operations on Generating Functions (6)

6. Convolution (product)

If $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(z)$, $\langle g_0, g_1, g_2, \ldots \rangle \longleftrightarrow G(z)$, and

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then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Proof:

$$\begin{aligned} F(z) \cdot G(z) &= (f_0 + f_1 z + f_2 z^2 + \ldots) \cdot (g_0 + g_1 z + g_2 z^2 + \ldots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0) z + (f_0 g_2 + f_1 g_1 + f_2 g_0) z^2 + \ldots \\ &\quad \text{by absolute convergence} \end{aligned}$$

Q.E.D.

Note that all terms involving the same power of z lie on a /sloped diagonal:

	$g_0 z^0$	$g_1 z^1$	$g_2 z^2$	$g_3 z^3$	$\ldots$
$f_0 z^0$	$f_0 g_0 z^0$	$f_0 g_1 z^1$	$f_0 g_2 z^2$	$f_0 g_3 z^3$	$\ldots$
$f_1 z^1$	$f_1 g_0 z^1$	$f_1 g_1 z^2$	$f_1 g_2 z^3$	$\ldots$	
$f_2 z^2$	$f_2 g_0 z^2$	$f_2 g_1 z^3$	$\ldots$		
$f_3 z^3$	$f_3 g_0 z^3$	$\ldots$			

Operations on Generating Functions (6)

6. Convolution (product)

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$$h_n = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0 = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=k} a_i b_j$$

then $\langle h_0, h_1, h_2, \ldots \rangle \longleftrightarrow F(z) \cdot G(z)$.

Example

$$\begin{aligned}
\langle 1,1,1,1,\ldots \rangle \cdot \left\langle 0,1,\frac{1}{2},\frac{1}{3},\ldots \right\rangle &= \left\langle 1\cdot 0, 1\cdot 0 + 1\cdot 1, 1\cdot 0 + 1\cdot 1 + 1\cdot\frac{1}{2}, 1\cdot 0 + 1\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{3}, \ldots \right\rangle \\
&= \left\langle 0, 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \ldots \right\rangle \\
&= \langle 0, H_1, H_2, H_3, \ldots \rangle
\end{aligned}$$

Hence:

$$\sum_{n\geqslant 1} H_n z^n = \frac{1}{1-z} \log \frac{1}{1-z}.$$

Example: the generating function of $g_n = n^2$

$$
\begin{aligned}
\langle 1,1,1,1,\ldots\rangle &\longleftrightarrow \frac{1}{1-z} \\
\langle 1,2,3,4,\ldots\rangle &\longleftrightarrow \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} \\
\langle 0,1,2,3,\ldots\rangle &\longleftrightarrow z\cdot\frac{1}{(1-z)^2} = \frac{z}{(1-z)^2} \\
\langle 1,4,9,16,\ldots\rangle &\longleftrightarrow \frac{d}{dz}\frac{z}{(1-z)^2} = \frac{1+z}{(1-z)^3} \\
\langle 0,1,4,9,\ldots\rangle &\longleftrightarrow z\cdot\frac{1+z}{(1-z)^3} = \frac{z(1+z)}{(1-z)^3}
\end{aligned}
$$

Next subsection

Counting with Generating Functions

Example: Choosing a k-subset of an n-set

The binomial theorem yields:

$$\left\langle \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots \right\rangle \longleftrightarrow \sum_{k \geqslant 0} \binom{n}{k} z^k = (1+z)^n$$

- The coefficient of z^k in $(1+z)^n$ is the number of ways to choose k distinct items from a set of size n.
- For example, the coefficient of z^2 is the number of ways to choose 2 items from a set with n elements.
- Similarly, the coefficient of z^{n+1} is the number of ways to choose $n+1$ items from a n-set, which is zero.

Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is:
$A(z) = \sum_{n \geqslant 0} [n \text{ can be selected}]\, z^n$

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

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Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

- If any even number of elements can be selected:

$$A(z) = 1 + z^2 + z^4 + z^6 + \cdots = \frac{1}{1-z^2}$$

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The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is:
$A(z) = \sum_{n\geqslant 0} [n \text{ can be selected}]\, z^n$

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

- If any even number of elements can be selected:

$$A(z) = 1 + z^2 + z^4 + z^6 + \cdots = \frac{1}{1-z^2}$$

- If any positive even number of elements can be selected:

$$A(x) = z^2 + z^4 + z^6 + \cdots = \frac{z^2}{1-z^2}$$

Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is:
$A(z) = \sum_{n \geqslant 0} [n \text{ can be selected}]\, z^n$

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

- If any number of elements multiple of 5 can be selected:

$$A(z) = 1 + z^5 + z^{10} + z^{15} + \cdots = \frac{1}{1-z^5}$$

Building Generating Functions that Count

The generating function for the number of ways to choose n elements from a 1-basket $\mathscr{A}$ (a (multi)set of identical elements) is:
$A(z) = \sum_{n \geqslant 0} [n \text{ can be selected}]\, z^n$

Examples of GF selecting items from a set $\mathscr{A}$:

- If any natural number of elements can be selected:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

- If at most four elements can be selected:

$$A(z) = 1 + z + z^2 + z^3 + z^4 = \frac{1-z^5}{1-z}$$

- If at most one element can be selected:

$$A(z) = \frac{1-z^2}{1-z} = 1 + z$$

Counting elements of two sets

Convolution Rule

Let $A(z)$ be the generating function for selecting an item from (multi)set $\mathscr{A}$, and let $B(z)$, be the generating function for selecting an item from (multi)set $\mathscr{B}$.
If $\mathscr{A}$ and $\mathscr{B}$ are disjoint, then the generating function for selecting items from the union $\mathscr{A} \cup \mathscr{B}$ is the product $A(z) \cdot B(z)$.

Proof. To count the number of ways to select n items from $\mathscr{A} \cup \mathscr{B}$ we have to select j items from $\mathscr{A}$ and $n-j$ items from $\mathscr{B}$, where $0 \leqslant j \leqslant n$.
Summing over all the possible values of j gives a total of

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

ways to select n items from $\mathscr{A} \cup \mathscr{B}$. This is precisely the coefficient of z^n in the series for $A(z) \cdot B(z)$ Q.E.D.

How many nonnegative integer solutions does the equation $x_1 + x_2 = n$ have?

- There is one way to solve the equation $x_1 = n$, so the generating function for the number of solutions of $x_1 = n$ is:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

- The same holds for $x_2 = n$.
- Then the generating function of the number of solutions of $x_1 + x_2 = n$ is the convolution of $1/(1-z)$ with itself:

$$\begin{aligned}
H(z) &= (1 + z + z^2 + z^3 + \cdots)(1 + z + z^2 + z^3 + \cdots) \\
&= (1 \cdot 1) + (z \cdot 1 + 1 \cdot z) + (1 \cdot z^2 + z \cdot z + z^2 \cdot 1) \\
&\quad + (1 \cdot z^3 + z \cdot z^2 + z^2 \cdot z + z^3 \cdot 1) + \ldots \\
&\quad \text{by absolute convergence} \\
&= 1 + 2z + 3z^2 + \ldots + (n+1)z^n + \ldots \\
&= \frac{1}{(1-z)^2}
\end{aligned}$$

How many nonnegative integer solutions does the equation $x_1 + x_2 = n$ have?

- There is one way to solve the equation $x_1 = n$, so the generating function for the number of solutions of $x_1 = n$ is:

$$A(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

- The same holds for $x_2 = n$.
- Then the generating function of the number of solutions of $x_1 + x_2 = n$ is the convolution of $1/(1-z)$ with itself:

$$\begin{aligned} H(z) &= (1 + z + z^2 + z^3 + \cdots)(1 + z + z^2 + z^3 + \cdots) \\ &= (1 \cdot 1) + (z \cdot 1 + 1 \cdot z) + (1 \cdot z^2 + z \cdot z + z^2 \cdot 1) \\ &\quad + (1 \cdot z^3 + z \cdot z^2 + z^2 \cdot z + z^3 \cdot 1) + \ldots \\ &\quad \text{by absolute convergence} \\ &= 1 + 2z + 3z^2 + \ldots + (n+1)z^n + \ldots \\ &= \frac{1}{(1-z)^2} \end{aligned}$$

Indeed, this equation has $n+1$ solutions:

$$0 + n, 1 + (n-1), 2 + (n-2) \ldots, (n-1) + 1, n + 0.$$

The number of integer solutions of the equation $x_1 + x_2 + \cdots + x_k = n$

Theorem

The number of ways to distribute n identical objects into k bins is $\binom{n+k-1}{k}$.

Proof:

- The generating function of the sequence of the number of solutions of $x_1 + \ldots + x_k = n$ is the convolution $1/(1-z)^k$ of k copies of $1/(1-z)$.
- But for an analytic function $f(z)$ in a neighborhood of the origin:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \ldots + \frac{f^{(n)}(0)}{n!}z^n + \ldots$$

- For $f(z) = \dfrac{1}{(1-z)^k}$ it is $f^{(n)}(z) = k(k+1)\cdots(k+n-1) \cdot \dfrac{1}{(1-z)^{k+n}}$, so:

$$\frac{f^{(n)}(0)}{n!} = \frac{k^{\overline{n}}}{n!} = \frac{(n+k-1)^{\underline{n}}}{n!} = \binom{n+k-1}{n}$$

A summary of properties of generating functions

Let $\langle g_0, g_1, g_2, \ldots \rangle$ and $\langle h_0, h_1, h_2, \ldots \rangle$ be sequences of complex numbers.
Let $G(z) = \sum_{n \geqslant 0} g_n z^n$ and $H(z) = \sum_{n \geqslant 0} h_n z^n$ be their generating functions.
The following operations are legitimate:

sequence	generic term	g.f.
$\langle \alpha g_0 + \beta h_0, \alpha g_1 + \beta h_1, \alpha g_2 + \beta h_2, \ldots \rangle$	$\alpha g_n + \beta h_n$	$\alpha G(z) + \beta H(z)$
$\langle 0, \ldots, 0, g_0, g_1, \ldots \rangle$	$g_{n-m}[n \geqslant m]$	$z^m G(z)$
$\langle g_m, g_{m+1}, g_{m+2}, \ldots \rangle$	g_{n+m}	$\dfrac{G(z) - \sum_{k=0}^{m-1} g_k z^k}{z^m}$
$\langle a_1, 2a_2, 3a_3, \ldots \rangle$	$(n+1)g_{n+1}$	$G'(z)$
$\langle 0, a_1, 2a_2, 3a_3, \ldots \rangle$	ng_n	$zG'(z)$
$\langle 0, a_0, \frac{a_1}{2}, \ldots \rangle$	$\frac{g_{n-1}}{n}[n > 0]$	$\int_0^z G(w)dw$
$\langle g_0 h_0, g_0 h_1 + g_1 h_0, g_0 h_2 + g_1 h_1 + g_2 h_0, \ldots \rangle$	$\sum_{k=0}^{n} g_k h_{n-k}$	$G(z) \cdot H(z)$

where:

- $\text{undefined} \cdot [\text{False}] = 0$; and
- $\int_0^z G(w)dw = \int_0^1 zG(tz)dt = \Gamma(z)$ where $\Gamma'(z) = G(z)$ and $\Gamma(0) = 0$.

Warmup: The old lady and her pets

The problem

When a certain old lady walks her pets, she brings:

- three, four, or five dogs;
- a cage with several pairs of rabbits;
- and (sometimes) her crocodile.

In how many ways can she walk n pets, for $n \geqslant 0$?

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Using generating functions

Let $D(z)$, $R(z)$, and $C(z)$ be the generating functions of the number of ways the old lady can walk dogs, rabbits, and crocodiles, respectively:

$$D(z) = z^3 + z^4 + z^5\,;\;\; R(z) = 1 + z^2 + z^4 + \cdots = \frac{1}{1-z^2}\,;\;\; C(z) = 1 + z$$

The generating function $A(z)$ of the number of ways the old lady can walk pets is thus:

$$A(z) = D(z) \cdot R(z) \cdot C(z) = \frac{z^3 + z^4 + z^5}{1-z}$$

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When a certain old lady walks her pets, she brings:

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- and (sometimes) her crocodile.

In how many ways can she walk n pets, for $n \geqslant 0$?

Solution

For $m \geqslant 0$ integer, $G(z) = z^m$ is the generating function of $g_n = [n = m]$.
$G(z) = (1-z)^{-1}$ is the generating function of $g_n = 1$.
Then for every $n \geqslant 0$, the number of ways the old lady can walk her pets is:

$$a_n = [z^n]A(z) = \sum_{m=3}^{5} \sum_{k=0}^{n} [k = m] = \sum_{m=3}^{5} [n \geqslant m]$$

For example, for $n = 6$ the old lady has three choices:

- three dogs, one pair of rabbits, and the crocodile;
- four dogs and one pair of rabbits;
- five dogs and the crocodile.

Derivatives of the generating function

Theorem

If $G(z) = \sum_{n \geqslant 0} g_n z^n$, then for every $k \geqslant 0$,

$$G^{(k)}(z) = \sum_{n \geqslant 0} (n+k)^{\underline{k}} g_{n+k} z^n$$

Derivatives of the generating function

Theorem

If $G(z) = \sum_{n \geqslant 0} g_n z^n$, then for every $k \geqslant 0$,

$$G^{(k)}(z) = \sum_{n \geqslant 0} (n+k)^{\underline{k}} g_{n+k} z^n$$

The thesis is true for $k = 0$ as $n^{\underline{0}}$ is an empty product.
If the thesis is true for k, then

$$\begin{aligned} G^{(k+1)}(z) &= \sum_{n \geqslant 0} (n+k)^{\underline{k}} n g_{n+k} z^{n-1} [n \geqslant 1] \\ &= \sum_{n \geqslant 0} (n+1+k)^{\underline{k}} (n+1)^{\underline{1}} g_{n+1+k} z^n \\ &= \sum_{n \geqslant 0} (n+1+k)^{\underline{k+1}} g_{n+1+k} z^n \end{aligned}$$

Derivatives of the generating function

Theorem

If $G(z) = \sum_{n \geqslant 0} g_n z^n$, then for every $k \geqslant 0$,

$$G^{(k)}(z) = \sum_{n \geqslant 0} (n+k)^{\underline{k}} g_{n+k} z^n$$

Corollary

For every $n \geqslant 0$,

$$g_n = \frac{n^{\underline{n}}}{n!} g_n = \frac{1}{n!} \sum_{n \geqslant 0} (n+k)^{\underline{n}} g_{n+k} 0^k = \frac{G^{(n)}(0)}{n!}$$

Distribute n objects into k bins so that there is at least one object in each bin

Theorem

The number of k-tuples of positive integers such that $x_1 + x_2 + \ldots + x_k = n$ is $\binom{n-1}{k-1}$.

Proof: (sketch)

- For $k = 1$ bin there is one way of distributing n objects if $n > 0$ and none if $n = 0$.
- Then the generating function of the sequence of the number of ways to put n objects in 1 bin is $C(z) = z + z^2 + z^3 + \ldots = z/(1-z)$.
- For $k \geqslant 1$ arbitrary, the generating function of the solution is the convolution of k copies of $C(z)$ with itself:

$$H(z) = (C(z))^k = \frac{z^k}{(1-z)^k} .$$

- But this is the shift by k positions to the right of $\dfrac{1}{(1-z)^k} = \sum_{n \geqslant 0} \binom{n+k-1}{n} z^n$, so:

$$H(z) = \sum_{n \geqslant 0} \binom{n+k-1}{n} z^{n+k} = \sum_{n \geqslant 0} \binom{n+k-1}{k-1} z^{n+k} = \sum_{n \geqslant k} \binom{n-1}{k-1} z^n .$$

Q.E.D.

Example: 100 Euros

In how many ways can 100 Euros be changed using smaller banknotes?

Generating functions for selecting banknotes of 5, 10, 20 or 50 Euros:

$$A(z) = z^0 + z^5 + z^{10} + z^{15} + \cdots = \frac{1}{1-z^5}$$

$$B(z) = z^0 + z^{10} + z^{20} + z^{30} + \cdots = \frac{1}{1-z^{10}}$$

$$C(z) = z^0 + z^{20} + z^{40} + z^{60} + \cdots = \frac{1}{1-z^{20}}$$

$$D(z) = z^0 + z^{50} + z^{100} + z^{150} + \cdots = \frac{1}{1-z^{50}}$$

Generating function for obtaining sums of euros using banknotes:

$$P(z) = A(z)B(z)C(z)D(z) = \frac{1}{(1-z^5)(1-z^{10})(1-z^{20})(1-z^{50})}$$

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Example: 100 Euro (2)

1. Observation:

$$
\begin{array}{rcl}
(1-z^5)(1 & + & z^5+\cdots+z^{45}+2z^{50}+2z^{55}+\cdots+2z^{95}+3z^{100}+3z^{105}+\cdots+3z^{145}+4z^{150}+\cdots)= \\
1 & + & z^5+\cdots+z^{45}+2z^{50}+2z^{55}+\cdots+2z^{95}+3z^{100}+3z^{105}+\cdots+3z^{145}+4z^{150}\cdots \\
 & - & z^5-\cdots-z^{45}-z^{50}-2z^{55}-\cdots-2z^{95}-2z^{100}-3z^{105}-\cdots-3z^{145}-3z^{150}-4z^{155}-\cdots \\
=1 & + & z^{50}+z^{100}+z^{150}+z^{200}+\cdots=\dfrac{1}{1-z^{50}}
\end{array}
$$

By dividing both sides by $1-z^5$ we get:

$$
F(z)=A(z)D(z)=\frac{1}{(1-z^5)(1-z^{50})}=\sum_{k\geqslant 0}\left(\left\lfloor\frac{k}{10}\right\rfloor+1\right)z^{5k}=\sum_{k\geqslant 0}f_k z^{5k}
$$

TAL TECH

Example: 100 Euro (2)

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$$
\begin{aligned}
(1-z^5)(1 &+ z^5+\cdots+z^{45}+2z^{50}+2z^{55}+\cdots+2z^{95}+3z^{100}+3z^{105}+\cdots+3z^{145}+4z^{150}+\cdots) = \\
1 &+ z^5+\cdots+z^{45}+2z^{50}+2z^{55}+\cdots+2z^{95}+3z^{100}+3z^{105}+\cdots+3z^{145}+4z^{150}\cdots \\
&- z^5-\cdots-z^{45}-z^{50}-2z^{55}-\cdots-2z^{95}-2z^{100}-3z^{105}-\cdots-3z^{145}-3z^{150}-4z^{155}-\cdots \\
=1 &+ z^{50}+z^{100}+z^{150}+z^{200}+\cdots = \frac{1}{1-z^{50}}
\end{aligned}
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By dividing both sides by $1-z^5$ we get:

$$F(z) = A(z)D(z) = \frac{1}{(1-z^5)(1-z^{50})} = \sum_{k\geqslant 0}\left(\left\lfloor\frac{k}{10}\right\rfloor+1\right)z^{5k} = \sum_{k\geqslant 0} f_k z^{5k}$$

2. Similarly:

$$G(z) = B(z)C(z) = \frac{1}{(1-z^{10})(1-z^{20})} = \sum_{k\geqslant 0}\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)z^{10k} = \sum_{k\geqslant 0} g_k z^{10k}$$

Example: 100 Euro (3)

- Convolution:

$$P(z) = F(z)G(z) = \sum_{n \geqslant 0} c_n z^{5n}$$

- The coefficient of z^{100} equals:

$$\begin{aligned}
c_{20} &= f_0 g_{10} + f_2 g_9 + f_4 g_8 + \cdots + f_{20} g_0 \\
&= \sum_{k=0}^{10} f_{2k} g_{10-k} \\
&= \sum_{k=0}^{10} \left(\left\lfloor \frac{2k}{10} \right\rfloor + 1 \right) \left(\left\lfloor \frac{10-k}{2} \right\rfloor + 1 \right) \\
&= \sum_{k=0}^{10} \left(\left\lfloor \frac{k+5}{5} \right\rfloor \right) \left(\left\lfloor \frac{12-k}{2} \right\rfloor \right) \\
&= 1 \cdot (6+5+5+4+4) + 2 \cdot (3+3+2+2+1) + 3 \cdot 1 \\
&= 24 + 22 + 3 = 49 \,.
\end{aligned}$$

TALTECH

Next subsection

Generating function for arbitrary binomial coefficients

Theorem (Generalized binomial theorem)

For every $r \in \mathbb{R}$,

$$(1+z)^r = \sum_{n \geqslant 0} \binom{r}{n} z^n$$

Generating function for arbitrary binomial coefficients

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Indeed, let $G(z) = (1+z)^r$ where $r \in \mathbb{R}$ is arbitrary:

- By differentiating $n \geqslant 0$ times, $G^{(n)}(z) = r \cdots (r-1) \cdots (r-n+1) \cdot (1+z)^{r-n}$.
- Then,

$$\frac{G^{(n)}(0)}{n!} = \frac{r^{\underline{n}}}{n!} = \binom{r}{n}$$

  .
- As n is arbitrary and the correspondence between sequences and generating functions is one-to-one, the thesis follows.

Generating function for arbitrary binomial coefficients

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.
- As n is arbitrary and the correspondence between sequences and generating functions is one-to-one, the thesis follows.

Example

$$\sqrt{1+z} = \sum_{n \geqslant 0} \binom{1/2}{n} z^n$$

TAL TECH

Vandermonde's identity

Theorem

For every $r, s \in \mathbb{C}$ and $n \geqslant 0$,

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k}$$

Vandermonde's identity

Theorem

For every $r, s \in \mathbb{C}$ and $n \geqslant 0$,

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k}$$

Proof:

$$\begin{aligned}
\sum_{n \geqslant 0} \binom{r+s}{n} z^n &= (1+z)^{r+s} \\
&= (1+z)^r \cdot (1+z)^s \\
&= \left(\sum_{n \geqslant 0} \binom{r}{n} z^n \right) \cdot \left(\sum_{n \geqslant 0} \binom{s}{n} z^n \right) \\
&= \sum_{n \geqslant 0} \left(\sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} \right) z^n ,
\end{aligned}$$

whence the thesis by uniqueness of coefficients.

Vandermonde's identity

Theorem

For every $r, s \in \mathbb{C}$ and $n \geqslant 0$,

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k}$$

Example: $r = 3$, $s = 6$

1
1 1
1 2 1
1 **3** **3** 1
1 4 6 4 1
1 5 10 10 5 1
1 **6** **15** 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 **36** 84 126 126 84 36 9 1

$$\binom{9}{2} = \binom{3}{0}\binom{6}{2} + \binom{3}{1}\binom{6}{1} + \binom{3}{2}\binom{6}{0}$$

Vandermonde's identity for $r = s = n$

Special case: $r = s = n$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k \geqslant 0} \binom{n}{k}^2$$

Example: $\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 = \binom{8}{4}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 **70** 56 28 8 1

1
1 1
1 4 1
1 9 9 1
1 **16** **36** **16** **1**

$1 + 16 + 36 + 16 + 1 = 70$

Sequence $\left\langle \binom{m}{0}, 0, -\binom{m}{1}, 0, \binom{m}{2}, 0, -\binom{m}{3}, 0, \binom{m}{4}, 0, \ldots, \right\rangle$

Let's take sequences

$$\left\langle \binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{n}, \ldots \right\rangle \longleftrightarrow F(z) = (1+z)^m$$

and

$$\left\langle \binom{m}{0}, -\binom{m}{1}, \binom{m}{2}, \ldots, (-1)^n \binom{m}{n}, \ldots \right\rangle \longleftrightarrow G(z) = F(-z) = (1-z)^m$$

Then the convolution corresponds to the function $(1+z)^m(1-z)^m = (1-z^2)^m$ that gives the identity of binomial coefficients:

$$\sum_{j=0}^{n} \binom{m}{j} \binom{m}{n-j} (-1)^j = (-1)^{\lfloor n/2 \rfloor} \binom{m}{\lfloor n/2 \rfloor} [n \text{ is even}]$$

Other useful binomial identities

Sign change and falling powers

$$(-1)^n r^{\underline{n}} = (n-r-1)^{\underline{n}} \ \forall r \in \mathbb{R} \ \forall n \geqslant 0$$

Proof: $(-1)^n \cdot r \cdot (r-1) \cdots (r-n+2) \cdot (r-n+1) = (n-r-1) \cdot (n-r-2) \cdots (1-r) \cdot (-r)$

Generating function for binomial coefficients with upper index increasing

For every $r \geqslant 0$,

$$\frac{1}{(1-z)^{r+1}} = \sum_{n \geqslant 0} (-1)^n \binom{-1-r}{n} z^n = \sum_{n \geqslant 0} \binom{r+n}{n} z^n$$

In addition, if $r = m$ is an integer,

$$\frac{1}{(1-z)^{m+1}} = \sum_{n \geqslant 0} \binom{m+n}{n} z^n = \sum_{n \geqslant 0} \binom{m+n}{m} z^n$$

and by shifting,

$$\frac{z^m}{(1-z)^{m+1}} = \sum_{n \geqslant 0} \binom{m+n}{m} z^{m+n} = \sum_{n \geqslant 0} \binom{n}{m} z^n$$

Generating functions cheat sheet

$$\frac{1}{1-z} = \sum_{n\geqslant 0} z^n$$

$$\frac{z}{(1-z)^2} = \sum_{n\geqslant 0} nz^n$$

$$(1+z)^r = \sum_{n\geqslant 0} \binom{r}{n} z^n, \quad r \in \mathbb{R}$$

$$\frac{1}{(1-z)^{r+1}} = \sum_{n\geqslant 0} \binom{r+n}{n} z^n, \quad r \in \mathbb{R}$$

$$\frac{z^m}{(1-z)^{m+1}} = \sum_{n\geqslant 0} \binom{n}{m} z^n, \quad m \in \mathbb{N}$$

$$\log \frac{1}{1-z} = \sum_{n\geqslant 1} \frac{z^n}{n}$$

$$\frac{1}{1-z} \log \frac{1}{1-z} = \sum_{n\geqslant 1} H_n z^n$$