Special Numbers

ITT9132 Concrete Mathematics

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Some Fibonacci Identities

Cassini's Identity $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ for all $n > 0$

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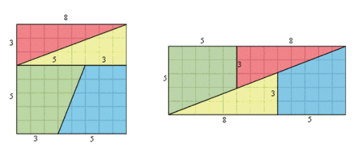
The Chessboard Paradox



Some Fibonacci Identities

Cassini's Identity $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for all $n > 0$

Divisors f_n and f_{n+1} are relatively prime and f_k divides f_{nk}:

$$\gcd(f_n, f_m) = f_{\gcd(n,m)}$$

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$$\gcd(f_n, f_m) = f_{\gcd(n,m)}$$

Matrix Calculus If A is the 2×2 matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}, \qquad \text{for } n > 0.$$

Note that this yields Cassini's identity, because $\det A = -1$.

Some Fibonacci Identities (2)

Fibonacci Numbers and Pascal's Triangle:

$$f_{n+1} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j}$$

n	f_n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$
0	**0**	1								
1	**1**	1	1							
2	**1**	1	2	1						
3	**2**	1	3	3	1					
4	**3**	1	4	6	4	1				
5	**5**	1	5	10	10	5	1			
6	**8**	1	6	15	20	15	6	1		
7	**13**	1	7	21	35	35	21	7	1	
8	**21**	1	8	28	56	70	56	28	8	1

Some Fibonacci Identities (3)

Continued fractions

The continued fraction composed entirely of 1s equals the ratio of successive Fibonacci numbers:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-2} + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} = \frac{f_{n+1}}{f_n},$$

where $a_1 = a_2 = \cdots = a_n = 1$.

For example

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{f_5}{f_4} = \frac{5}{3} = 1.(6)$$

Some Fibonacci Identities (3)

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The continued fraction composed entirely of 1s equals the ratio of successive Fibonacci numbers:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \over a_{n-2} + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = \frac{f_{n+1}}{f_n},$$

where $a_1 = a_2 = \cdots = a_n = 1$.

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Some applications of Fibonacci numbers (1)

Let S_n denote the number of subsets of $\{1, 2, \dots, n\}$ that do not contain consecutive elements.
For example, when $n = 3$ the "good" subsets are $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}$: hence, $S_3 = 5$.

Theorem

For every $n \geqslant 1$, $S_n = f_{n+2}$.

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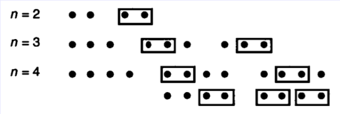
For every $n \geqslant 1$, $S_n = f_{n+2}$.

Proof:

- We can identify a subset A of $\{1, 2, \dots, n\}$ with a binary word w of n bits, such that $i \in A$ if and only if the ith bit of w is 1.
- Then A has no consecutive elements if and only if 11 does not appear in w.
- For $n = 1$ both 0 and 1 are "good", so $S_1 = 2 = f_3$.
  For $n = 2$, 00, 01 and 10 are all "good", but 11 is "bad". Thus, $S_2 = 3 = f_4$
- For $n \geqslant 3$, a "good" word w of length n must be either $w = u0$ where u is a "good" word of length $n - 1$, or $w = v01$ where v is a "good" word of length $n - 2$.
  Hence, $S_n = S_{n-1} + S_{n-2}$ for every $n \geqslant 3$.
- Then $S_n = f_{n+2}$ for every $n \geqslant 1$.

Some applications of Fibonacci numbers (2)

Draw n dots in a line. If each domino can cover exactly two such dots, in how many ways can (non-overlapping) dominoes be placed?
For example:



Theorem

The number of possible placements of dominoes with n dots is $D_n = f_{n+1}$.

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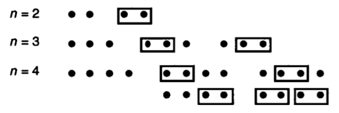

Theorem

The number of possible placements of dominoes with n dots is $D_n = f_{n+1}$.

Proof:

- Consider the rightmost dot in any such placement P.
- If this dot is not covered by a domino, then P minus the last dot determines a solution for $n-1$ dots.
- If the last dot is covered by a domino, then the last two dots in P are covered by this domino. Removing this rightmost domino then gives a solution for $n-2$ dots.
- Hence, $D_n = D_{n-1} + D_{n-2}$ for every $n \geqslant 3$.
- As $D_2 = 2 = f_3$ (no dominos, one domino) and $D_3 = 3 = f_4$, the thesis follows.

Some applications of Fibonacci numbers (3)

An ordered composition of a positive integer n is a sum of the form $a_1 + \ldots + a_k = n$, where $a_1, \ldots, a_k$ are positive integers and the order of the summands is taken into account.
For example:

- $4 = 1+3 = 3+1 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$
- $5 = 4+1 = 1+4 = 1+1+3 = 1+3+1 = 3+1+1 = \ldots = 1+1+1+1+1.$

Theorem

- The number of ordered compositions of the positive integer n into odd summands is $T_n = f_n$.
- The number of ordered compositions of the positive integer n where all summands are 1 or 2 is $B_n = f_{n+1}$.

Fibonacci number system

Zeckendorf's theorem

Every integer $n \geqslant 2$ has a unique writing

$$n = f_{k_1} + f_{k_2} + \ldots + f_{k_r}$$

such that:

1. $k_1 > k_2 > \ldots > k_r > 1$, and
2. no two k_is are consecutive.

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such that:

1. $k_1 > k_2 > \ldots > k_r > 1$, and
2. no two k_is are consecutive.

Proof:

- The thesis is true for $1 = f_2$, $2 = f_3$, $3 = f_4$ and $4 = 3 + 1 = f_4 + f_2$.
- Suppose the thesis is true for every positive $m < n$.
- Let k_1 be the largest such that $f_{k_1} \leqslant n$. If $f_{k_1} = n$ we are done.
- Otherwise, let $n' = n - f_{k_1} > 0$. If $n' = 1$ we let $k_2 = 2$, and we are done.
- Otherwise, $n' \geqslant 2$, so by induction $n' = f_{k_2} + \ldots + f_{k_r}$ in a unique way under conditions 1 and 2.
- But it cannot be $k_2 = k_1 - 1$, otherwise we would have chosen $f_{k_1+1} = f_{k_1} + f_{k_2}$ when taking the largest Fibonacci number not larger than n. Hence, the writing $n = f_{k_1} + f_{k_2} + \ldots + f_{k_r}$ is also unique.

Approximations

Observation

$$\lim_{n\to\infty} \widehat{\Phi}^n = 0$$

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- $f_n \asymp \frac{\Phi^n}{\sqrt{5}}$ as $n \to \infty$
- $f_n = \left\lfloor \frac{\Phi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor$

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For example:

$$f_{10} = \left\lfloor \frac{\Phi^{10}}{\sqrt{5}} + \frac{1}{2} \right\rfloor = \left\lfloor 55.00364\ldots + \frac{1}{2} \right\rfloor = \lfloor 55.50364\ldots \rfloor = 55$$

$$f_{11} = \left\lfloor \frac{\Phi^{11}}{\sqrt{5}} + \frac{1}{2} \right\rfloor = \left\lfloor 88.99775\ldots + \frac{1}{2} \right\rfloor = \lfloor 89.49775\ldots \rfloor = 89$$

Approximations

Observation

$$\lim_{n\to\infty} \widehat{\Phi}^n = 0$$

- $f_n \asymp \frac{\Phi^n}{\sqrt{5}}$ as $n \to \infty$
- $f_n = \left\lfloor \frac{\Phi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor$
- $\frac{f_n}{f_{n-1}} \to \Phi$ as $n \to \infty$

For example:

$$\frac{f_{11}}{f_{10}} = \frac{89}{55} \approx 1.61818182 \approx \Phi = 1.61803\ldots$$

Fibonacci numbers with negative index: Idea

Question

What can f_n be when n is a negative integer?

We want the basic properties to be satisfied for every $n \in \mathbb{Z}$:

- Defining formula:
$$f_n = f_{n-1} + f_{n-2}\,.$$
- Expression by golden ratio:
$$f_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right).$$
- Matrix form:
$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Consequently, Cassini's identity too.)

Note: For $n = 0$, the above suggest $f_{-1} = 1$ ...

Fibonacci numbers with negative index: Formula

Theorem

For every $n \geqslant 1$,

$$f_{-n} = (-1)^{n-1} f_n$$

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Proof: As $(1-\phi z)\cdot(1-\hat{\phi}z) = 1 - z - z^2$, it is $\phi^{-1} = -\hat{\phi} = 0.618\ldots$
Then for every $n \geqslant 1$,

$$\begin{aligned} f_{-n} &= \frac{1}{\sqrt{5}}\left(\phi^{-n} - \hat{\phi}^{-n}\right) \\ &= \frac{1}{\sqrt{5}}\left((-\hat{\phi})^n - (-\phi)^n\right) \\ &= \frac{(-1)^{n+1}}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \\ &= (-1)^{n-1} f_n , \end{aligned}$$

Q.E.D.

Fibonacci numbers with negative index: Formula

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Q.E.D.

Another proof is by induction with the defining relation in the form $f_{n-2} = f_n - f_{n-1}$, with initial conditions $f_1 = 1$, $f_0 = 0$.

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_{n+1} f_k + f_n f_{k-1}$$

Warmup: The generalized Cassini's identity

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For every $n, k \in \mathbb{Z}$,

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Why generalized?

Because for $k = 1 - n$ we get

$$f_1 = (-1)^{n-2} f_{n-1} f_{n+1} + (-1)^{n-1} f_n^2 \,,$$

which is Cassini's identity multiplied by $(-1)^n$.

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_{n+1} f_k + f_n f_{k-1}$$

Proof:

- Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_{n+1} f_k + f_n f_{k-1}$$

Proof:

- Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. We know that $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$ for every $n \geqslant 0$.
- But $B = A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} f_0 & f_{-1} \\ f_{-1} & f_{-2} \end{pmatrix}$ satisfies the same recurrence with negative indices:

$$\begin{aligned} B^n \cdot B &= \begin{pmatrix} f_{-n+1} & f_{-n} \\ f_{-n} & f_{-n-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} f_{-n} & f_{-n+1} - f_{-n} \\ f_{-n-1} & f_{-n} - f_{-n-1} \end{pmatrix} \\ &= \begin{pmatrix} f_{-(n+1)+1} & f_{-(n+1)} \\ f_{-(n+1)} & f_{-(n+1)-1} \end{pmatrix} \end{aligned}$$

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_{n+1} f_k + f_n f_{k-1}$$

Proof:

- Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$ for every $n \in \mathbb{Z}$.
- By associativity of matrix product, $A^{n+k} = A^n \cdot A^k$ for every $n, k \in \mathbb{Z}$: that is,

$$\begin{pmatrix} f_{n+k+1} & f_{n+k} \\ f_{n+k} & f_{n+k-1} \end{pmatrix} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \cdot \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$$

- The thesis then follows by comparing the elements in the upper right corner.

Warmup: The generalized Cassini's identity

Theorem

For every $n, k \in \mathbb{Z}$,

$$f_{n+k} = f_{n+1} f_k + f_n f_{k-1}$$

Alternative proof by induction:

- For every $n \in \mathbb{Z}$ let $P(n)$ be the following proposition:

$$\forall k \in \mathbb{Z} \,.\, f_{n+k} = f_k f_{n+1} + f_{k-1} f_n \,.$$

- For $n = 0$ we get $f_k = f_k \cdot 1 + 0$.
  For $n = 1$ we get $f_{k+1} = f_k \cdot 1 + f_{k-1} \cdot 1$.
- If $n \geqslant 2$ and $P(n-1)$ and $P(n-2)$ hold, then:

$$\begin{aligned} f_{n+k} &= f_{n-1+k} + f_{n-2+k} \\ &= f_k f_n + f_{k-1} f_{n-1} + f_k f_{n-1} + f_{k-1} f_{n-2} \\ &= f_k f_{n+1} + f_{k-1} f_n \,. \end{aligned}$$

- If $n < 0$ and $P(n+1)$ and $P(n+2)$ hold, then

$$\begin{aligned} f_{n+k} &= f_{n+2-k} - f_{n+1-k} \\ &= f_k f_{n+3} + f_{k-1} f_{n+2} - f_k f_{n+2} - f_{k-1} f_{n+1} \\ &= f_k f_{n+1} + f_{k-1} f_n \,. \end{aligned}$$

A note on generating functions for bi-infinite sequences

Question

Can we define f_n for every $n \in \mathbb{Z}$ via a single power series which depends from both positive and negative powers of the variable?
(We can renounce such $G(z)$ to be defined in $z = 0$.)

A note on generating functions for bi-infinite sequences

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Can we define f_n for every $n \in \mathbb{Z}$ via a single power series which depends from both positive and negative powers of the variable?
(We can renounce such $G(z)$ to be defined in $z = 0$.)

Answer: Yes, but it would not be practical!

A generalization of Laurent's theorem goes as follows:
Let f be an analytic function defined in an annulus $A = \{z \in \mathbb{C} \mid r < |z| < R\}$.
Then there exists a bi-infinite sequence $\langle a_n \rangle_{n \in \mathbb{Z}}$ such that:

1. the series $\sum_{n \geqslant 0} a_n z^n$ has convergence radius $\geqslant R$;
2. the series $\sum_{n \geqslant 1} a_{-n} z^n$ has convergence radius $\geqslant 1/r$;
3. for every $z \in A$ it is $\sum_{n \in \mathbb{Z}} a_n z^n = f(z)$.

We could set $r = 0$, but the power series $\sum_{n \geqslant 1} a_{-n} z^n$ would then need to have infinite convergence radius! (i.e., $\lim_{n \to \infty} \sqrt[n]{|a_{-n}|} = 0$.) However, $\lim_{n \to \infty} \sqrt[n]{f_n} = \phi$.
Also, the intersection of two annuli can be empty: making controls on feasibility of operations much more difficult to check. (Not so for "disks with a hole in zero".)

Fibonacci numbers cheat sheet

- Recurrence:

$$\begin{aligned} f_0 &= 0\,;\; f_1 = 1\,; \\ f_n &= f_{n-1} + f_{n-2} \qquad \forall n \geqslant 2\,; \\ f_{-n} &= (-1)^{n-1} f_n \qquad \forall n > 0\,. \end{aligned}$$

- Generating function:

$$\sum_{n \geqslant 0} f_n z^n = \frac{z}{1 - z - z^2} \;\; \forall z \in \mathbb{C},\, |z| < \frac{1}{\phi}\,.$$

- Matrix form:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \;\; \forall n \in \mathbb{Z}.$$

- Generalized Cassini's identity:

$$f_{n+k} = f_k f_{n+1} + f_{k-1} f_n \;\; \forall n, k \in \mathbb{Z}.$$

- Greatest common divisor:

$$\gcd(f_m, f_n) = f_{\gcd(m,n)} \;\; \forall m, n \in \mathbb{Z}.$$

TAL TECH

Next section

Next subsection

Harmonic numbers

Definition

The harmonic numbers are given by the formula

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{for } n \geqslant 0, \text{ with } H_0 = 0$$

- H_n is the discrete analogue of the natural logarithm.
- The first twelve harmonic numbers are shown in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11
H_n	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$	$\frac{83711}{27720}$

The graph of $f(n) = H_n$

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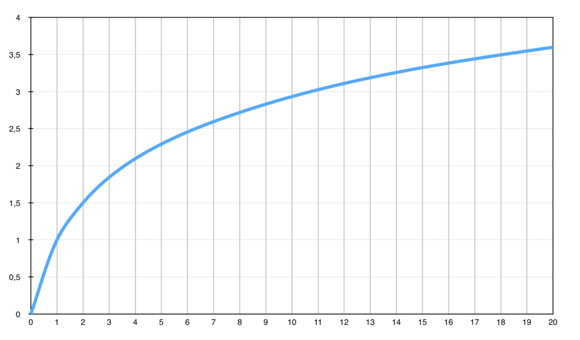

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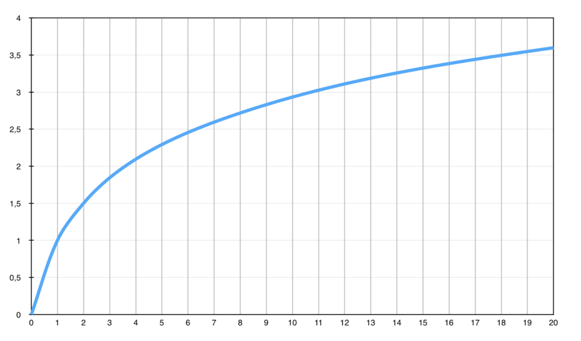

Looks a bit like the graph of the logarithm . . .

Harmonic numbers and binary logarithms

Theorem

For every positive integer n:

$$1+\frac{1}{2}\lfloor \lg n \rfloor \leqslant H_n \leqslant 1+\lfloor \lg n \rfloor$$

Proof:

- Let $m=\lfloor \lg n \rfloor$ be the unique natural number such that $2^m \leqslant n \leqslant 2^{m+1}-1$.
- Then $H_{2^m} \leqslant H_n \leqslant H_{2^{m+1}-1}$, that is:

$$1+\sum_{k=0}^{m-1}\sum_{j=2^k+1}^{2^{k+1}}\frac{1}{2^{k+1}} \leqslant 1+\sum_{k=0}^{m-1}\sum_{j=2^k+1}^{2^{k+1}}\frac{1}{j} \leqslant H_n \leqslant \sum_{k=0}^{m}\sum_{j=2^k}^{2^{k+1}-1}\frac{1}{j} \leqslant \sum_{k=0}^{m}\sum_{j=2^k}^{2^{k+1}-1}\frac{1}{2^k}$$

- Clearly, the left-hand side is $1+\sum_{k=0}^{m-1}\frac{1}{2}=1+\frac{m}{2}$ and the right-hand side is $\sum_{k=0}^{m}1=1+m$. Q.E.D.

TAL TECH

Harmonic numbers and natural logarithms

Theorem

For every positive integer n:

$$\ln n < H_n < 1 + \ln n$$

Proof:

- First, let $f(x) = \frac{1}{n}\,[n < x \leqslant n+1]$ for $x > 1$.
- Then $f(x) > 1/x$ for every $x > 1$, so:

$$H_n = \int_1^{n+1} f(x)\,dx > \int_1^n \frac{dx}{x} = \ln n\,.$$

- Now, let $g(x) = \frac{1}{n}\,[n-1 \leqslant x < n]$ for $x > 0$.
- Then $g(x) < 1/x$ for every $x > 0$, so:

$$H_n = 1 + \int_1^n g(x)\,dx < 1 + \int_1^n \frac{dx}{x} = 1 + \ln n\,.$$

A card trick: Formulation

The problem

We have a deck of cards, and want to stack them on a table so that:

1. the stack hangs as much as possible out of the table;
2. the edge of the cards is parallel to that of the table; and
3. the stack does not fall down, according to the law of gravity.

Question:

What is the maximum overhang that we can reach?
(provided we have enough many cards)

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What is the maximum overhang that we can reach?
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Solution:

The stack can overhang by as much as we want!
(provided we have enough many cards)

A card trick: Experiment

With one card:

- We can put the card so that its center of gravity is precisely on the edge of the table.
- Let's call this overhang an overhang unit, so that a card is 2 overhang units long.

With two cards:

- We count cards from top to bottom, rather than from bottom to top.
- We put the second card so that it hangs by half a unit over the table, and the first card to that it hangs by one unit over the first card.
- Then the center of gravity of the stack is precisely on the edge of the table.

With three cards:

- We put the third card so that it hangs by one third of a unit over the table.
- We put the second card so that it hangs by half a unit over the third card.
- We put the first card to that it hangs by one unit over the second card.
- Then the center of gravity of the stack is precisely on the edge of the table.

A card trick: General idea and solution

Given n cards in the stack, we count the topmost at first, and identify the table with an $n+1$st card.

- Call d_k the overhang of the first card over the kth, so $d_1 = 0$.
  For example, with $n = 3$ we had $d_2 = 1$, $d_3 = 3/2$, and $d_4 = 11/6$ was the overhang over the table.
- If we want that the center of gravity of the entire stack is on the edge of the table, we must also have the center of gravity of the first k cards over the edge of the $k+1$st card. Then:

$$d_{k+1} = \frac{(d_1+1)+(d_2+1)+\ldots+(d_k+1)}{k} \quad \text{for every } 1 \leqslant k \leqslant n$$

- By multiplying by k and writing for two consecutive values, we have:

$$\begin{aligned} kd_{k+1} &= k + d_1 + d_2 + \ldots + d_k \\ (k-1)d_k &= k - 1 + d_1 + d_2 + \ldots + d_{k-1} \end{aligned}$$

  and by subtracting,

$$kd_{k+1} - (k-1)d_k = 1 + d_k$$

A card trick: General idea and solution

Given n cards in the stack, we count the topmost at first, and identify the table with an $n+1$st card.

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  For example, with $n = 3$ we had $d_2 = 1$, $d_3 = 3/2$, and $d_4 = 11/6$ was the overhang over the table.
- If we want that the center of gravity of the entire stack is on the edge of the table, we must also have the center of gravity of the first k cards over the edge of the $k+1$st card.
- We have thus found that d_k must satisfy:

$$
\begin{aligned}
d_1 &= 0, \\
kd_{k+1} &= (k-1)d_k + 1 + d_k = kd_k + 1 \text{ for every } k \geqslant 1 .
\end{aligned}
$$

- But the recurrence $d_{k+1} = d_k + \frac{1}{k}$ with the initial condition $d_1 = 0$ has the solution:

$$d_{k+1} = H_k \text{ for every } k \geqslant 0 .$$

  This is also the maximum possible overhang with k cards, because as soon as we move a card far from the edge of the table, the stack topples.

Generating function of harmonic numbers

Theorem

$$\sum_{n\geqslant 0} H_n z^n = \frac{1}{1-z} \ln \frac{1}{1-z}$$

Indeed, $\frac{1}{1-z} = \sum_{n\geqslant 0} z^n$, $\ln \frac{1}{1-z} = \sum_{n\geqslant 0} \frac{1}{n} [n \geqslant 1] z^n$, and

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=0}^{n} \frac{1}{k} [k \geqslant 1] 1^{n-k}$$

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$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=0}^{n} \frac{1}{k} [k \geqslant 1] 1^{n-k}$$

A general remark

If $G(z)$ is the generating function of the sequence $\langle g_0, g_1, g_2, \ldots \rangle$, then $G(z)/(1-z)$ is the generating function of the sequence of the prefix sums of the original sequence:

$$\text{if } G(z) = \sum_{n \geqslant 0} g_n z^n \text{ then } \frac{G(z)}{1-z} = \sum_{n \geqslant 0} \left(\sum_{k=0}^{n} g_k \right) z^n$$

Harmonic numbers of higher order

Definition

For $n \geqslant 1$ and $m \geqslant 2$ integer, the nth harmonic number of order m is

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$$

As with the "first order" harmonic numbers, we put $H_0^{(m)} = 0$ as an empty sum.

For $m \geqslant 2$ the quantities

$$H_\infty^{(m)} = \lim_{n \to \infty} H_n^{(m)}$$

exist finite: they are the values of the *Riemann zeta function*

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}\,, \;\; s > 1$$

for $s = m$.

Euler's γ constant

Euler's approximation of harmonic numbers

For every $n \geqslant 1$,

$$H_n - \ln n = 1 - \sum_{m \geqslant 2} \frac{1}{m} \left(H_n^{(m)} - 1 \right)$$

Euler's γ constant

Euler's approximation of harmonic numbers

For every $n \geqslant 1$,

$$H_n - \ln n = 1 - \sum_{m \geqslant 2} \frac{1}{m} \left(H_n^{(m)} - 1 \right)$$

For $k \geqslant 2$ we can write:

$$\ln \frac{k}{k-1} = \ln \frac{1}{1 - \frac{1}{k}} = \sum_{m \geqslant 1} \frac{1}{m \cdot k^m}$$

As $\ln(a/b) = \ln a - \ln b$ and $\ln 1 = 0$, by summing for k from 2 to n we get:

$$\ln n = \sum_{k=2}^{n} \sum_{m \geqslant 1} \frac{1}{m \cdot k^m} = \sum_{m \geqslant 1} \sum_{k=2}^{n} \frac{1}{m \cdot k^m} = H_n - 1 + \sum_{m \geqslant 2} \left(H_n^{(m)} - 1 \right)$$

Euler's γ constant

Euler's approximation of harmonic numbers

For every $n \geqslant 1$,

$$H_n - \ln n = 1 - \sum_{m \geqslant 2} \frac{1}{m} \left(H_n^{(m)} - 1 \right)$$

For $m \geqslant 2$, $H_n^{(m)}$ converges from below to $\zeta(m)$.

It turns out that $\zeta(s) - 1 \sim 2^{-s}$, therefore the series $\sum_{m \geqslant 2} \frac{1}{m} (\zeta(m) - 1)$ converges.

The quantity

$$\gamma = 1 - \sum_{m \geqslant 2} \frac{1}{m} (\zeta(m) - 1)$$

is called Euler's constant. The following approximation holds:

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o\left(\frac{1}{n^3} \right)$$

Next subsection

Harmonic numbers

Properties:

- Harmonic and Stirling cyclic numbers: $H_n = \frac{1}{n!}\left[{n+1 \atop 2}\right]$ for every $n \geqslant 1$;
- $\sum_{k=1}^{n} H_k = (n+1)(H_{n+1} - 1)$ for every $n \geqslant 1$;
- $\sum_{k=1}^{n} kH_k = \binom{n+1}{2}\left(H_{n+1} - \frac{1}{2}\right)$ for every $n \geqslant 1$;
- $\sum_{k=1}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right)$ for every $n \geqslant 1$;
- $\lim_{n\to\infty} H_n = \infty$;
- $H_n \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4}$ where $\gamma \approx 0.57721\,56649\,01533$ denotes *Euler's constant*.

Approximation

- $H_{10} \approx 2.92896\,82578\,96$
- $H_{1000000} \approx 14.39272\,67228\,65723\,63138\,11275$

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Harmonic numbers

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Harmonic numbers and binomial coefficients

Theorem

$$\sum_{k=0}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$$

Harmonic numbers and binomial coefficients

Theorem

$$\sum_{k=0}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$$

Proof:

- For $v(x) = \binom{x}{m+1}$ it is $\Delta v(x) = \binom{x+1}{m+1} - \binom{x}{m+1} = \binom{x}{m}$
- Summing by parts with $u(x) = H_x$:

$$\begin{aligned}
\sum_0^{n+1} \binom{x}{m} H_x \, \delta x &= \binom{x}{m+1} H_x \Big|_0^{n+1} - \sum_0^{n+1} \binom{x+1}{m+1} x^{\underline{-1}} \, \delta x \\
&= \binom{n+1}{m+1} H_{n+1} - \frac{1}{m+1} \sum_0^{n+1} \binom{x}{m} \, \delta x \\
&= \binom{n+1}{m+1} H_{n+1} - \frac{1}{m+1} \binom{x}{m+1} \Big|_0^{n+1} \\
&= \binom{n+1}{m+1} H_{n+1} - \frac{1}{m+1} \binom{n+1}{m+1}
\end{aligned}$$

Q.E.D.

Harmonic numbers and binomial coefficients

Theorem

$$\sum_{k=0}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$$

Corollary

For $m = 0$ we get:

$$\sum_{k=0}^{n} H_k = (n+1)(H_{n+1} - 1) = (n+1)H_n - n$$

For $m = 1$ we get:

$$\sum_{k=0}^{n} kH_k = \frac{n(n+1)}{2} \left(H_{n+1} - \frac{1}{2} \right) = \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4}$$

Sum of averaged harmonic numbers

Theorem

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2 + H_n^{(2)}}{2} .$$

Proof:

- Let $S_n = \sum_{k=1}^{n} \frac{H_k}{k}$. Then, as $H_k = \sum_{j=1}^{k} \frac{1}{j}$, we have:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} = \sum_{1 \leqslant j \leqslant k \leqslant n} \frac{1}{jk} \\ &= \frac{1}{2} \left(\sum_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant n} \frac{1}{jk} + \sum_{k=1}^{n} \left(\frac{1}{k} \right)^2 \right) \\ &= \frac{1}{2} \left(\left(\sum_{j=1}^{n} \frac{1}{j} \right) \cdot \left(\sum_{k=1}^{n} \frac{1}{k} \right) + \sum_{k=1}^{n} \frac{1}{k^2} \right) \\ &= \frac{1}{2} \left(H_n^2 + H_n^{(2)} \right) \end{aligned}$$

Q.E.D.

Next section

Bernoulli numbers: History

Jakob Bernoulli (1654-1705) worked on the functions:

$$S_m(n) = 0^m + 1^m + \ldots + (n-1)^m = \sum_{k=0}^{n-1} k^m = \sum_0^n x^m \delta x$$

Plotting an expansion with respect to n yields:

$$\begin{array}{lclllllll}
S_0(n) & = & n & & & & & \\
S_1(n) & = & \frac{1}{2}n^2 & -\frac{1}{2}n & & & & \\
S_2(n) & = & \frac{1}{3}n^3 & -\frac{1}{2}n^2 & +\frac{1}{6}n & & & \\
S_3(n) & = & \frac{1}{4}n^4 & -\frac{1}{2}n^3 & +\frac{1}{4}n^2 & & & \\
S_4(n) & = & \frac{1}{5}n^5 & -\frac{1}{2}n^4 & +\frac{1}{3}n^3 & -\frac{1}{30}n & & \\
S_5(n) & = & \frac{1}{6}n^6 & -\frac{1}{2}n^5 & +\frac{5}{12}n^4 & -\frac{1}{12}n^2 & & \\
S_6(n) & = & \frac{1}{7}n^7 & -\frac{1}{2}n^6 & +\frac{1}{2}n^5 & -\frac{1}{6}n^3 & +\frac{1}{42}n & \\
S_7(n) & = & \frac{1}{8}n^8 & -\frac{1}{2}n^7 & +\frac{7}{12}n^6 & -\frac{7}{24}n^4 & +\frac{1}{12}n^2 & \\
S_8(n) & = & \frac{1}{9}n^9 & -\frac{1}{2}n^8 & +\frac{2}{3}n^7 & -\frac{7}{15}n^5 & +\frac{2}{9}n^3 & -\frac{1}{30}n \\
S_9(n) & = & \frac{1}{10}n^{10} & -\frac{1}{2}n^9 & +\frac{3}{4}n^8 & -\frac{7}{10}n^6 & +\frac{1}{2}n^4 & -\frac{3}{20}n^2
\end{array}$$

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Bernoulli observed the following regularities:

- The leading coefficient of S_m is always $\frac{1}{m+1} = \frac{1}{m+1}\binom{m+1}{0}$.
- The coefficient of n^m in S_m is always $-\frac{1}{2} = -\frac{1}{2} \cdot \frac{1}{m+1} \cdot \binom{m+1}{1}$.
- The coefficient of n^{m-1} in S_m is always $\frac{m}{12} = \frac{1}{6} \cdot \frac{1}{m+1} \cdot \binom{m+1}{2}$.
- The coefficient of n^{m-2} in S_m is always 0.
- The coefficient of n^{m-3} in S_m is always $-\frac{m(m-1)(m-2)}{720} = -\frac{1}{30} \cdot \frac{1}{m+1} \cdot \binom{m+1}{4}$.
- The coefficient of n^{m-4} in S_m is always 0.
- The coefficient of n^{m-5} in S_m is always $\frac{1}{42} \cdot \frac{1}{m+1} \cdot \binom{m+1}{6}$.
- And so on, and so on ...

Bernoulli numbers

Definition

The kth Bernoulli number is the unique value B_k such that, for every $m \geqslant 0$,

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

As $S_m(1) = 0^m = [m=0]$, we can also use the following recurrence:

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = [m=0]$$

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Examples:

$$\begin{array}{lll}
m=0 & B_0 = 1 & \\
m=1 & B_0 + 2B_1 = 0 & B_1 = -\frac{1}{2} \\
m=2 & B_0 + 3B_1 + 3B_2 = 0 & B_2 = \frac{1}{6} \\
m=3 & B_0 + 4B_1 + 6B_2 + 4B_3 = 0 & B_3 = 0 \\
m=4 & B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0 & B_4 = -\frac{1}{30}
\end{array}$$

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

Bernoulli numbers and the Riemann zeta function

Theorem

For every $n \geqslant 1$,

$$B_n = -n\zeta(1-n)$$

Theorem

For every $n \geqslant 1$,

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}$$

In particular,

$$\zeta(2) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

From this and Stirling's approximation we get:

$$|B_{2n}| \sim 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n} \text{ for } n \to \infty$$

Generating function of the Bernoulli numbers . . . *almost*

Because of the approximation in the previous slide,

$$\limsup_{n \geqslant 0} \sqrt[n]{|B_n|} = \limsup_{n \geqslant 0} \frac{n}{2\pi e} = +\infty,$$

and the Bernoulli numbers do not have a generating function.

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However:

$$\sum_{n \geqslant 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1} .$$

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and the Bernoulli numbers do not have a generating function.
However:

$$\sum_{n \geqslant 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}.$$

This function is analytic in a neighborhood of the origin because of:

Singularity removal theorem

Let $f(z)$ be analytic in the open disk $D_r(c)$ of center c and radius r, except at most the center c itself.
If $f(z)$ is bounded in $D_r(c) \setminus \{c\}$, then it can be extended to an analytic function in the entire $D_r(c)$.

In particular, if $\lim_{z \to c} f(z)$ exists, then $f(z)$ has an analytic continuation.
This is the case of $\dfrac{z}{e^z - 1}$, which converges to 1 for $z \to 0$.