Generating Functions

ITT9132 Concrete Mathematics
Lecture 13 – 22 April 2019

Contents

Generating Functions

Three basic principles:

1. Every sequence $\langle g_n \rangle_{n \geqslant 0}$ of complex numbers such that $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} < \infty$ has a generating function
$$G(z) = \sum_{n \geqslant 0} g_n z^n \tag{1}$$
which is analytic in a neighborhood of the origin of the complex plane.
2. Vice versa, every function $G(z)$ which is analytic in a neighborhood of the origin of the complex plane is the generating function of the sequence
$$g_n = [z^n]G(z) = \frac{G^{(n)}(0)}{n!} \, . \tag{2}$$
3. The correspondence is one-to-one: two analytic functions which coincide in a neighborhood of the origin identify the same sequence, and vice versa, each sequence such that $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} < \infty$ identifies a unique analytic function in a neighborhood of the origin.

Given a closed form for $G(z)$, we will see how to:

- Determine a closed form for g_n.
- Compute infinite sums.
- Solve recurrence equations.

When convenient, we will sum over all integers, under the tacit assumption that:

$$g_n = 0 \text{ whenever } n < 0.$$

A counterexample to point 3 for functions of a real variable

For every $x \in \mathbb{R}$ let:

$$f(x) = e^{-1/x^2} \cdot [x \neq 0] \, .$$

- f is infinitely differentiable at every $x \neq 0$.
- For $x = 0$ we have:

  $$\lim_{x \to 0} f(x) = \lim_{t \to +\infty} e^{-t} = 0 \, ,$$

  so f is continuous in the origin (recall our convention that $\mathrm{undefined} \cdot [\mathrm{False}] = 0$).
- Now, every derivative of f has the form $f^{(n)}(x) = p(1/x) \cdot e^{-1/x^2}$ for some polynomial p, so its limit in $x = 0$ is 0.
- Then extending $f^{(n)}(x)$ at $x = 0$ by setting $f^{(n)}(x) = 0$ turns f into a *smooth* function, which has derivatives of any order.
- But this also means that the Taylor series of f at $x = 0$ is identically zero.
- If point 3 of the previous slide held for functions of a real variable, then f should be identically zero in a neighborhood of 0: which is not the case.

Generating function manipulations

Let $F(z)$ and $G(z)$ be the generating functions for the sequences $\langle f_n \rangle$ and $\langle g_n \rangle$.

We put $f_n = g_n = 0$ for every $n < 0$, and $\text{undefined} \cdot 0 = 0$.

- $\alpha F(z) + \beta G(z) = \sum_n (\alpha f_n + \beta g_n) z^n$, $\qquad \alpha, \beta \in \mathbb{C}$
- $z^m G(z) = \sum_n g_{n-m} z^n$, $\qquad$ integer $m \geqslant 0$
- $\frac{G(z) - \sum_{k=0}^{m-1} g_k z^k}{z^m} = \sum_n g_{n+m} [n \geqslant 0] z^n$, $\qquad$ integer $m \geqslant 0$
- $G(cz) = \sum_n c^n g_n z^n$, $\qquad c \in \mathbb{C}$
- $G'(z) = \sum_n (n+1) g_{n+1} z^n$
- $zG'(z) = \sum_n n g_n z^n$
- $F(z)G(z) = \sum_n \left(\sum_k f_k g_{n-k} \right) z^n$, $\qquad$ in particular, $\frac{G(z)}{1-z} = \sum_n \left(\sum_{0 \leqslant k \leqslant n} g_k \right) z^n$
- $\int_0^z G(w) \mathrm{d}w = \sum_{n \geqslant 1} \frac{1}{n} g_{n-1} z^n$, $\qquad$ where $\int_0^z G(w) \mathrm{d}w = z \int_0^1 G(zt) \mathrm{d}t$

Basic sequences and their generating functions

For $m \geqslant 0$ integer

- $\langle 1,0,0,0,0,0,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}[n=0]\,z^n = 1$
- $\langle 0,\dots,0,1,0,0,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}[n=m]\,z^n = z^m$
- $\langle 1,1,1,1,1,1,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0} z^n = \frac{1}{1-z}$
- $\langle 1,-1,1,-1,1,-1,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}(-1)^n z^n = \frac{1}{1+z}$
- $\langle 1,0,1,0,1,0,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}[2|n]\,z^n = \frac{1}{1-z^2}$
- $\langle 1,0,\dots,0,1,0,\dots,0,1,0,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}[m|n]\,z^n = \frac{1}{1-z^m}$
- $\langle 1,2,3,4,5,6,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}(n+1)z^n = \frac{1}{(1-z)^2}$
- $\langle 1,2,4,8,16,32,\dots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}2^n z^n = \frac{1}{1-2z}$

Basic sequences and their generating functions (2)

For $m \geqslant 0$ integer and for $c \in \mathbb{C}$

- $\langle 1,4,6,4,1,0,0,\ldots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}\binom{4}{n}z^n = (1+z)^4$
- $\left\langle 1,c,\binom{c}{2},\binom{c}{3},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}\binom{c}{n}z^n = (1+z)^c$
- $\left\langle 1,c,\binom{c+1}{2},\binom{c+2}{3},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}\binom{c+n-1}{n}z^n = \frac{1}{(1-z)^c}$
- $\langle 1,c,c^2,c^3,\ldots\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}c^n z^n = \frac{1}{1-cz}$
- $\left\langle 1,\binom{m+1}{m},\binom{m+2}{m},\binom{m+3}{m},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}\binom{m+n}{m}z^n = \frac{1}{(1-z)^{m+1}}$
- $\left\langle 0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 1}\frac{1}{n}z^n = \ln\frac{1}{1-z}$
- $\left\langle 0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}z^n = \ln(1+z)$
- $\left\langle 1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120},\ldots\right\rangle \quad \leftrightarrow \quad \sum_{n\geqslant 0}\frac{1}{n!}z^n = e^z$

Warmup: A simple generating function

Problem

Determine the generating function $G(z)$ of the sequence

$$g_n = 2^n + 3^n \, , n \geqslant 0$$

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Solution

- For $\alpha \in \mathbb{C}$, the generationg function of $\langle \alpha^n \rangle_{n \geqslant 0}$ is $G_\alpha(z) = \frac{1}{1 - \alpha z}$.
- By linearity, we get

$$G(z) = G_2(z) + G_3(z) = \frac{1}{1-2z} + \frac{1}{1-3z} \, .$$

TAL TECH

Extracting the even- or odd-numbered terms of a sequence

Let $\langle g_0, g_1, g_2, \ldots \rangle \leftrightarrow G(z)$.

Then

$$G(z) + G(-z) = \sum_n g_n (1 + (-1)^n) z^n = 2 \sum_n g_n [n \text{ is even}] z^n$$

Therefore

$$\langle g_0, 0, g_2, 0, g_4, \ldots \rangle \leftrightarrow \frac{G(z) + G(-z)}{2} = \sum_n g_{2n} z^{2n}$$

Similarly

$$\langle 0, g_1, 0, g_3, 0, g_5, \ldots \rangle \leftrightarrow \frac{G(z) - G(-z)}{2} = \sum_n g_{2n+1} z^{2n+1}$$

$$\langle g_0, g_2, g_4, \ldots \rangle \leftrightarrow \sum_n g_{2n} z^n$$

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Extracting the even- or odd-numbered terms of a sequence (2)

Example: $\langle 1,0,1,0,1,0,\ldots\rangle \leftrightarrow F(z) = \frac{1}{1-z^2}$

We have

$$\langle 1,1,1,1,1,\ldots\rangle \leftrightarrow G(z) = \frac{1}{1-z}.$$

Then the generating function for $\langle 1,0,1,0,1,0,\ldots\rangle$ is

$$\frac{1}{2}(G(z)+G(-z)) = \frac{1}{2}\left(\frac{1}{1-z}+\frac{1}{1+z}\right) = \frac{1}{2}\cdot\frac{1+z+1-z}{(1-z)(1+z)} = \frac{1}{1-z^2}$$

Extracting the even- or odd-numbered terms of a sequence (3)

Example: $\langle 0, 1, 3, 8, 21, \ldots \rangle = \langle f_0, f_2, f_4, f_6, f_8, \ldots \rangle$

We know that

$$\langle 0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots \rangle \leftrightarrow F(z) = \frac{z}{1 - z - z^2}.$$

Then the generating function for $\langle f_0, 0, f_2, 0, f_4, 0, \ldots \rangle$ is

$$\begin{aligned}
\sum_n f_{2n} z^{2n} &= \frac{1}{2}\left(\frac{z}{1 - z - z^2} + \frac{-z}{1 + z - z^2} \right) \\
&= \frac{1}{2} \cdot \frac{z + z^2 - z^3 - z + z^2 + z^3}{(1 - z^2)^2 - z^2} \\
&= \frac{z^2}{1 - 3z^2 + z^4}
\end{aligned}$$

This gives

$$\langle 0, 1, 3, 8, 21, \ldots \rangle \ \leftrightarrow \ \sum_n f_{2n}\, z^n = \frac{z}{1 - 3z + z^2}$$

Next section

Solving recurrences

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n in terms of n.

"Algorithm"

1. Write down a single equation that expresses g_n in terms of other elements of the sequence. This equation should be valid for all integers n, assuming that $g_{-1} = g_{-2} = \cdots = 0$.
2. Multiply both sides of the equation by z^n and sum over all n. This gives, on the left, the sum $\sum_n g_n z^n$, which is the generating function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
3. Solve the resulting equation, getting a closed form for $G(z)$.
4. Expand $G(z)$ into a power series and read off the coefficient of z^n; this is a closed form for g_n.

Next subsection

Example: Fibonacci numbers revisited

Step 1 The recurrence

$$g_n = \begin{cases} 0, & \text{if } n \leqslant 0; \\ 1, & \text{if } n = 1; \\ g_{n-1} + g_{n-2} & \text{if } n > 1; \end{cases}$$

can be represented by the single equation

$$g_n = g_{n-1} + g_{n-2} + [n = 1],$$

where $n \in (-\infty, +\infty)$.
This is because the "simple" Fibonacci recurrence $g_n = g_{n-1} + g_{n-2}$ holds for every $n \geqslant 2$ by construction, and for every $n \leqslant 0$ as by hypothesis $g_n = 0$ if $n < 0$; but for $n = 1$ the left-hand side is 1 and the right-hand side is 0, so we need the correction summand $[n = 1]$.

Example: Fibonacci numbers revisited (2)

Step 2 For any n, multiply both sides of the equation by z^n ...

$$\begin{aligned}
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
g_{-2}z^{-2} &= g_{-3}z^{-2} + g_{-4}z^{-2} + [-2=1]z^{-2} \\
g_{-1}z^{-1} &= g_{-2}z^{-1} + g_{-3}z^{-1} + [-1=1]z^{-1} \\
g_0 &= g_{-1} + g_{-2} + [0=1] \\
g_1 z &= g_0 z + g_{-1} z + [1=1]z \\
g_2 z^2 &= g_1 z^2 + g_0 z^2 + [2=1]z^2 \\
g_3 z^3 &= g_2 z^3 + g_1 z^3 + [3=1]z^3 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

... and sum over all n.

$$\sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n=1] z^n$$

Example: Fibonacci numbers revisited (3)

Step 3 Write down $G(z) = \sum_n g_n z^n$ and transform

$$G(z) = \sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n=1] z^n =$$
$$= \sum_n g_n z^{n+1} + \sum_n g_n z^{n+2} + z =$$
$$= zG(z) + z^2 G(z) + z$$

Solving the equation yields

$$G(z) = \frac{z}{1 - z - z^2}$$

Step 4 Expansion the equation into power series $G(z) = \sum g_n z^n$ gives us the solution (see next slides):

$$\Phi^n - \widehat{\Phi}^n$$

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Next subsection

TAL TECH

Linear recurrences

Definition

A linear recurrence of order k is a recurrence of the form:

$$g_n = \beta_1 g_{n-1} + \ldots + \beta_k g_{n-k} + f(n) . \quad (3)$$

The associate homogeneous recurrence of (3) is the recurrence

$$g_n = \beta_1 g_{n-1} + \ldots + \beta_k g_{n-k} ,$$

that is, the recurrence with $f(n) = 0$ for every $n \geqslant 0$.

The Fundamental Theorem of Linear Recurrences

Theorem

The solution of a linear recurrence with given initial conditions is the sum of:

- the solution of the associate homogeneous recurrence with the given initial conditions, and
- the solution of the given linear recurrence with initial conditions equal to zero.

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Indeed, if:

$$h_0 = a_0, \ldots, h_{k-1} = a_{k-1}; \; h_n = \beta_1 h_{n-1} + \ldots + \beta_k h_{n-k} \text{ for all } n \geqslant k, \text{ and}$$
$$s_0 = 0, \ldots, s_{k-1} = 0; \; s_n = \beta_1 s_{n-1} + \ldots + \beta_k s_{n-k} + f(n) \text{ for all } n \geqslant k,$$

and $g_n = h_n + s_n$ for every $n \geqslant 0$, then:

- for $0 \leqslant n < k$ it is $g_n = a_n + 0 = a_n$;
- and for $n \geqslant k$ it is:

$$\begin{aligned} g_n &= \beta_1 h_{n-1} + \ldots + \beta_k h_{n-k} + \beta_1 s_{n-1} + \ldots + \beta_k s_{n-k} + f(n) \\ &= \beta_1 (h_{n-1} + s_{n-1}) + \ldots + \beta_k (h_{n-k} + s_{n-k}) + f(n) \\ &= \beta_1 g_{n-1} + \ldots + \beta_k g_{n-k} + f(n). \end{aligned}$$

Q.E.D.

Next section

Motivation

- A generating function is often in the form of a rational function

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials.

- Our goal is to find "partial fraction expansion" of $R(z)$, i.e. represent $R(z)$ in the form

$$R(z) = S(z) + T(z),$$

where $S(z)$ has known expansion into the power series, and $T(z)$ is a polynomial.

- A good candidate for $S(z)$ is a finite sum of functions of the form:

$$S(z) = \frac{a_1}{(1-\rho_1 z)^{m_1+1}} + \frac{a_2}{(1-\rho_2 z)^{m_2+1}} + \cdots + \frac{a_\ell}{(1-\rho_\ell z)^{m_\ell+1}},.$$

- We have proven the relation

$$\frac{1}{(1-\rho z)^{m+1}} = \sum_{n\geqslant 0} \binom{m+n}{m} \rho^n z^n$$

- Hence, the coefficient of z^n in expansion of $S(z)$ is:

$$s_n = a_1 \binom{m_1+n}{m_1} \rho_1^n + a_2 \binom{m_2+n}{m_2} \rho_2^n + \cdots + a_\ell \binom{m_\ell+n}{m_\ell} \rho_\ell^n.$$

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Step 1: Finding $\rho_1, \rho_2, \dots, \rho_m$

- Suppose $Q(z)$ has the form

$$Q(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_m z^m, \qquad \text{where } q_m \neq 0.$$

- The "reflected" polynomial Q^R has a relation to Q:

$$\begin{aligned} Q^R(z) &= z^m + q_1 z^{m-1} + q_2 z^{m-2} + \cdots + q_{m-1} z + q_m \\ &= z^m \left(1 + q_1 \frac{1}{z} + q_2 \frac{1}{z^2} + \cdots + q_{m-1} \frac{1}{z^{m-1}} + q_m \frac{1}{z^m}\right) \\ &= z^m Q\left(\frac{1}{z}\right) \end{aligned}$$

- If $\rho_1, \rho_2, \dots, \rho_m$ are roots of Q^R, then $(z - \rho_i) | Q^R(z)$:

$$Q^R(z) = (z - \rho_1)(z - \rho_2) \cdots (z - \rho_m)$$

- Then $(1 - \rho_i z) | Q(z)$:

$$Q(z) = z^m \left(\frac{1}{z} - \rho_1\right)\left(\frac{1}{z} - \rho_2\right) \cdots \left(\frac{1}{z} - \rho_m\right) = (1 - \rho_1 z)(1 - \rho_2 z) \cdots (1 - \rho_m z)$$

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TAL TECH

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$$Q^R(z) = (z - \rho_1)(z - \rho_2) \cdots (z - \rho_m)$$

- Then $(1 - \rho_i z) | Q(z)$:

$$Q(z) = z^m (\frac{1}{z} - \rho_1)(\frac{1}{z} - \rho_2) \cdots (\frac{1}{z} - \rho_m) = (1 - \rho_1 z)(1 - \rho_2 z) \cdots (1 - \rho_m z)$$

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- Suppose $Q(z)$ has the form

$$Q(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_m z^m, \qquad \text{where } q_m \neq 0.$$

- The "reflected" polynomial Q^R has a relation to Q:

$$\begin{aligned} Q^R(z) &= z^m + q_1 z^{m-1} + q_2 z^{m-2} + \cdots + q_{m-1} z + q_m \\ &= z^m \left(1 + q_1 \frac{1}{z} + q_2 \frac{1}{z^2} + \cdots + q_{m-1} \frac{1}{z^{m-1}} + q_m \frac{1}{z^m}\right) \\ &= z^m Q\left(\frac{1}{z}\right) \end{aligned}$$

- If $\rho_1, \rho_2, \ldots, \rho_m$ are roots of Q^R, then $(z - \rho_i) | Q^R(z)$:

$$Q^R(z) = (z - \rho_1)(z - \rho_2) \cdots (z - \rho_m)$$

- Then $(1 - \rho_i z) | Q(z)$:

$$Q(z) = z^m (\frac{1}{z} - \rho_1)(\frac{1}{z} - \rho_2) \cdots (\frac{1}{z} - \rho_m) = (1 - \rho_1 z)(1 - \rho_2 z) \cdots (1 - \rho_m z)$$

Step 1: Finding $\rho_1, \rho_2, \ldots, \rho_m$ (2)

In all, we have proven

Lemma

$Q^R(z) = (z-\rho_1)(z-\rho_2)\cdots(z-\rho_m)$ iff $Q(z) = (1-\rho_1 z)(1-\rho_2 z)\cdots(1-\rho_m z)$

Example: $Q(z) = 1 - z - z^2$

$$Q^R(z) = z^2 - z - 1$$

This $Q^R(z)$ has roots

$$z_1 = \frac{1+\sqrt{5}}{2} = \Phi \qquad \text{and} \qquad z_2 = \frac{1-\sqrt{5}}{2} = \widehat{\Phi}$$

Therefore $Q^R(z) = (z-\Phi)(z-\widehat{\Phi})$ and $Q(z) = (1-\Phi z)(1-\widehat{\Phi} z)$.

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Next subsection

Step 2: Decomposition into Partial Fractions

If the following conditions are valid for the fraction $\frac{P(z)}{Q(z)}$:

- all roots of $Q^R(z)$ are distinct (we denote these roots as $\rho_1, \rho_2, \ldots$),
- $\deg P(z) < \deg Q(z) = \ell$,

then the denominator is factorizable as $Q(z) = a_0(1 - \rho_1 z) \cdots (1 - \rho_\ell z)$ and the fraction can be expanded as:

$$\frac{P(z)}{Q(z)} = \frac{A_1}{1 - \rho_1 z} + \frac{A_2}{1 - \rho_1 z} + \cdots + \frac{A_\ell}{1 - \rho_\ell z}, \tag{4}$$

where $A_1, A_2, \ldots, A_\ell$ are constants.

The constants $A_1, A_2, \ldots, A_\ell$ can be found as a solution of the system of linear equations defined by the equality (4).

Example: Decomposition of $\frac{z^2-3z+28}{6z^3-5z^2-2z+1}$

- We have here $P(z) = z^2 - 3z + 28$ and $Q(z) = 6z^3 - 5z^2 - 2z + 1$;
- The reflected polynomial is $Q^R(z) = z^3 - 2z^3 - 5z + 6 = (z-1)(z+2)(z-3)$, so $Q(z) = (1-z)(1+2z)(1-3z)$.

Hence,

$$\frac{P(z)}{Q(z)} = \frac{A}{1-z} + \frac{B}{1+2z} + \frac{C}{1-3z} =$$

$$= \frac{A(1+2z)(1-3z) + B(1-z)(1-3z) + C(1-z)(1+2z)}{Q(z)} =$$

$$= \frac{(-6A+3B-2C)z^2 + (-A-4B+C)z + (A+B+C)}{Q(z)}$$

Comparing the numerator of this fraction with the polynomial $P_1(z)$ leads to the system of equations:

$$\begin{cases} -6A+3B-2C & = 1 \\ -A-4B+C & = -3 \\ A+B+C & = 28 \end{cases}$$

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Example $\frac{z^2-3z+28}{6z^3-5z^2-2z+1}$ (continuation)

The solution of the system is

$$A = -\frac{13}{3}, \quad B = \frac{119}{15}, \quad C = \frac{122}{5}.$$

So, we have

$$S(z) = \frac{-13}{3(1-z)} + \frac{119}{15(1+2z)} + \frac{122}{5(1-3z)}.$$

and the power series $S(z) = \sum_{n \geqslant 0} s_n z^n$, where:

$$s_n = -\frac{13}{3} + \frac{119}{15}(-2)^n + \frac{122}{5}3^n.$$

Next subsection

Step 2 (alternative): Partial Rational Expansion

Theorem 1 (for Distinct Roots)

If $R(z) = P(z)/Q(z)$ is the generating function for the sequence $\langle r_n \rangle$,
where $Q(z) = (1-\rho_1 z)(1-\rho_2 z)\cdots(1-\rho_\ell z)$,
and the numbers $(\rho_1, \ldots, \rho_\ell)$ are distinct,
and if $P(z)$ is a polynomial of degree less than ℓ, then

$$r_n = a_1\rho_1^n + a_2\rho_2^n + \cdots + a_\ell\rho_\ell^n, \qquad \text{where} \qquad a_k = \frac{-\rho_k P(1/\rho_k)}{Q'(1/\rho_k)}$$

Sketch of proof.

- We show that $R(z) = S(z)$ for $S(z) = \frac{a_1}{1-\rho_1 z} + \cdots + \frac{a_\ell}{1-\rho_\ell z}$ and any $z \neq \alpha_k = 1/\rho_k$ (only the points where $R(z)$ might be equal to infinity).
- L'Hôpital's Rule is used

continues ...

l'Hôpital's Rule for functions of one complex variable

With a small abuse of language, we say that $\lim_{z \to a} f(z) = \infty$ if $\lim_{z \to a} |f(z)| = +\infty$.

Theorem

Let f and g be differentiable in an open neighborhood of $a \in \mathbb{C}$, except at most a itself. If either:

- $\lim_{z \to a} f(z) = \lim_{z \to a} g(z) = 0$, or
- $\lim_{z \to a} |f(z)| = \lim_{z \to a} |g(z)| = \infty$,

then:

$$\lim_{z \to a} \frac{f(z)}{g(z)} \text{ and } \lim_{z \to a} \frac{f'(z)}{g'(z)}$$

both exist, finite or infinite, and are equal.

Step 2: Partial Rational Expansion (2)

Continuation of the proof.

- $T(z) = R(z) - S(z)$ is a rational function of z and it suffices to show that $\lim_{z\to\alpha_k}(z-\alpha_k)T(z) = 0$.
- Thus we need to prove the following equality

$$\lim_{z\to\alpha_k}(z-\alpha_k)R(z) = \lim_{z\to\alpha_k}(z-\alpha_k)S(z).$$

- Due to

$$\frac{a_k(z-\alpha_k)}{1-\rho_j z} = \frac{a_k(z-\frac{1}{\rho_k})}{1-\rho_j z} = \frac{-a_k(1-\rho_k z)}{\rho_k(1-\rho_j z)} \to 0, \text{ if } k \neq j \text{ and } z \to \alpha_k$$

the right-hand side is

$$\lim_{z\to\alpha_k}(z-\alpha_k)S(z) = \lim_{z\to\alpha_k}(z-\alpha_k)\frac{a_k(z-\alpha_k)}{1-\rho_k z} = \frac{-a_k}{\rho_k} = \frac{P(1/\rho_k)}{Q'(1/\rho_k)}$$

continues ...

Step 2: Partial Rational Expansion (3)

Continuation of the proof.

- The left-hand side limit is

$$\lim_{z\to\alpha_k}(z-\alpha_k)R(z) = \lim_{z\to\alpha_k}(z-\alpha_k)\frac{P(z)}{Q(z)} = P(\alpha_k)\lim_{z\to\alpha_k}\frac{z-\alpha_k}{Q(z)} = \frac{P(\alpha_k)}{Q'(\alpha_k)} = \frac{P(1/\rho_k)}{Q'(1/\rho_k)}$$

  by l'Hôpital's rule

- Alternatively, as $\alpha_k = 1/\rho_k$ is a root of $Q(z)$:

$$\lim_{z\to\alpha_k}(z-\alpha_k)\frac{P(z)}{Q(z)} = P(\alpha_k)\lim_{z\to\alpha_k}\frac{z-\alpha_k}{Q(z)-Q(\alpha_k)} = P(1/\rho_k)\cdot\frac{1}{Q'(1/\rho_k)},$$

Q.E.D.

General Expansion Theorem for Rational Generating Functions.

Theorem 2 (for possibly Multiple Roots)

If $R(z) = P(z)/Q(z)$ is the generating function for the sequence $\langle r_n \rangle$, where $Q(z) = (1-\rho_1 z)^{d_1} \cdots (1-\rho_\ell z)^{d_\ell}$ and the numbers $\rho_1, \ldots, \rho_\ell$ are distinct, and if $P(z)$ is a polynomial of degree less than $d = d_1 + \ldots + d_\ell$, then

$$r_n = f_1(n)\rho_1^n + \cdots + f_\ell(n)\rho_\ell^n, \qquad \text{for all} \qquad n \geqslant 0,$$

where each $f_k(n)$ is a polynomial of degree $d_k - 1$ with leading coefficient

$$a_k = \frac{(-\rho_k)^{d_k} P(1/\rho_k) d_k}{Q^{(d_k)}(1/\rho_k)} = \frac{P(1/\rho_k)}{(d_k-1)! \prod_{j \neq k} (1-\rho_j/\rho_k)^{d_j}}$$

Proof: (omitted) by induction on $d = d_1 + \ldots + d_\ell$.

Warmup: What if $\deg P \geqslant \deg Q$?

The problem

- The hypotheses of the Rational Expansion Theorem include that the degree of the numerator be smaller than that of the denominator.
- What if it is not so?

Warmup: What if $\deg P \geqslant \deg Q$?

The problem

- The hypotheses of the Rational Expansion Theorem include that the degree of the numerator be smaller than that of the denominator.
- What if it is not so?

Answer: It is a false problem!

If $\deg P \geqslant \deg Q$, then we can do *polynomial division* and uniquely determine two polynomials $S(z)$, $R(z)$ such that:

- $\deg R < \deg Q$; and
- $P(z) = Q(z) \cdot S(z) + R(z)$.

Then

$$\frac{P(z)}{Q(z)} = S(z) + \frac{R(z)}{Q(z)} :$$

the first summand only influences finitely many coefficients, and on the second one the Rational Expansion Theorem can be applied.

Example: Fibonacci numbers revisited once more(2)

Step 3 Solving the equation

$$G(z) = \frac{z}{1 - z - z^2}$$

Step 4 Expand the (rational) equation $G(z) = P(z)/Q(z)$ for $P(z) = z$ and $Q(z) = 1 - z - z^2$:

- From the example above we know that $Q(z) = (1 - \Phi z)(1 - \widehat{\Phi} z)$
- As $Q'(z) = -1 - 2z$, we have

$$\frac{-\Phi P(1/\Phi)}{Q'(1/\Phi)} = \frac{-1}{-1 - 2/\Phi} = \frac{\Phi}{\Phi + 2} = \frac{1}{\sqrt{5}}$$

and

$$\frac{-\widehat{\Phi} P(1/\widehat{\Phi})}{Q'(1/\widehat{\Phi})} = \frac{\widehat{\Phi}}{\widehat{\Phi} + 2} = -\frac{1}{\sqrt{5}}$$

- Theorem 1 gives us

$$g_n = \frac{\Phi^n - \widehat{\Phi}^n}{\sqrt{5}}$$

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