Generating Functions

ITT9132 Concrete Mathematics

Lecture 14 – 29 April 2019

Contents

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Solving recurrences

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n in terms of n.

"Algorithm"

1. Write down a single equation that expresses g_n in terms of other elements of the sequence. This equation should be valid for all integers n, assuming that $g_{-1} = g_{-2} = \cdots = 0$.
2. Multiply both sides of the equation by z^n and sum over all n. This gives, on the left, the sum $\sum_n g_n z^n$, which is the generating function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
3. Solve the resulting equation, getting a closed form for $G(z)$.
4. Expand $G(z)$ into a power series and read off the coefficient of z^n; this is a closed form for g_n.

Next subsection

Example: A more-or-less random recurrence.

Solve the following recurrence:

$$g_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } 0 \leqslant n < 2; \\ g_{n-1} + 2g_{n-2} + (-1)^n & \text{if } 2 \leqslant n; \end{cases}$$

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Step 1 Write the recurrence for every $n \in \mathbb{Z}$, taking into account the initial conditions:

$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n [n \geqslant 2] + a_0 [n = 0] + a_1 [n = 1]$$

For $n = 0$ it is $1 = 0 + 0 + a_0 [n = 0]$, so $a_0 = 1 = (-1)^0$.
For $n = 1$ it is $1 = 1 + 0 + a_1 [n = 1] = 1 + (-1)^1 + 1$, so $a_1 = 1$ and we can add $(-1)^n$ for $n = 1$ too.
The recurrence can then be represented by the single equation:

$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n [n \geqslant 0] + [n = 1] \,.$$

Some values:

n	0	1	2	3	4	5	6	7
g_n	1	1	4	5	14	23	52	97

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Example: A more-or-less random recurrence (2)

Step 2 Write down $G(z) = \sum_n g_n z^n$ and transform

$$G(z) = \sum_n g_n z^n = \sum_n g_{n-1} z^n + 2\sum_n g_{n-2} z^n + \sum_{n\geqslant 0} (-1)^n z^n + \sum_n [n=1] z^n$$
$$= \sum_n g_n z^{n+1} + 2\sum_n g_n z^{n+2} + \frac{1}{1+z} + z$$
$$= zG(z) + 2z^2 G(z) + \frac{1+z+z^2}{1+z}$$

Step 3 Solving the equation

$$G(z) = \frac{1+z+z^2}{(1-z-2z^2)(1+z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$$

Example: A more-or-less random recurrence (2)

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Example: A more-or-less random recurrence (3)

Step 4 Expand the (rational) equation $G(z) = P(z)/Q(z)$ for $P(z) = 1 + z + z^2$ and $Q(z) = (1-2z)(1+z)^2$:

- Theorem 2 gives us for some constant c:

$$g_n = a_1 2^n + (a_2 n + c)(-1)^n,$$

where

$$a_1 = \frac{P(1/2)}{0!(1+1/2)^2} = \frac{4(1+1/2+1/4)}{9} = \frac{7}{9}$$

and

$$a_2 = \frac{P(-1)}{1!(1+2)} = \frac{1+1-1}{3} = \frac{1}{3}$$

- Special case $n = 0$ implies $1 = g_0 = \frac{7}{9} + c$ that gives $c = 1 - \frac{7}{9} = \frac{2}{9}$.
- The answer is

$$g_n = \frac{7}{9} 2^n + \left(\frac{1}{3} n + \frac{2}{9} \right) (-1)^n.$$

If $R(z) = P(z)/Q(z)$ the generating function for the sequence $\langle r_n \rangle$, where $Q(z) = (1-\rho_1 z)^{d_1} \cdots (1-\rho_\ell z)^{d_\ell}$ and the numbers $(\rho_1, \ldots, \rho_\ell)$ are distinct, and if $P(z)$ is a polynomial of degree less than $d_1 + \ldots + d_\ell$, then

$$r_n = f_1(n)\rho_1^n + \cdots + f_\ell(n)\rho_\ell^n, \qquad \text{for all} \qquad n \geqslant 0,$$

where each $f_k(n)$ is a polynomial of degree $d_k - 1$ with a leading coefficient

$$a_k = \frac{(-\rho_k)^{d_k} P(1/\rho_k) d_k}{Q^{(d_k)}(1/\rho_k)} = \frac{P(1/\rho_k)}{(d_k-1)! \prod_{j \neq k} (1 - \rho_j/\rho_k)^{d_j}}$$

Decomposition into Partial Fractions

The same function: $G(z) = \frac{P(z)}{Q(z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$

- Decompose it as

$$G(z) = \frac{A}{1-2z} + \frac{B}{1+z} + \frac{C}{(1+z)^2}$$

- Expand

$$\begin{aligned} G(z) &= \frac{A}{1-2z} + \frac{B}{1+z} + \frac{C}{(1+z)^2} = \\ &= \frac{A(1+z)^2 + B(1-2z)(1+z) + C(1-2z)}{(1-2z)(1+z)^2} = \\ &= \frac{(A-2B)z^2 + (2A-B-2C)z + A + B + C}{(1-2z)(1+z)^2} \end{aligned}$$

continues ...

Decomposition into Partial Fractions (2)

The function: $G(z) = \frac{P(z)}{Q(z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$

- System of equations:

$$\begin{cases} A - 2B & = 1 \\ 2A - B - 2C & = 1 \\ A + B + C & = 1 \end{cases}$$

- The solution: $A = \frac{7}{9}, B = -\frac{1}{9}, C = \frac{1}{3}$
- The result of decomposition $G(z) = \frac{7}{9(1-2z)} - \frac{1}{9(1+z)} + \frac{1}{3(1+z)^2}$
- using the basic identity

$$\frac{a}{(1-\rho z)^k} = \sum_{n\geqslant 0} \binom{n+k-1}{k-1} a\rho^n z^n,$$

we get the power series

$$G(z) = \sum_{n\geqslant 0} \left[\frac{7}{9} 2^n - \frac{1}{9}(-1)^n + \frac{n+1}{3}(-1)^n \right] z^n = \sum_{n\geqslant 0} g_n z^n,$$

where

$$g_n = \frac{7}{9} 2^n + \left(\frac{1}{3} n + \frac{2}{9} \right) (-1)^n.$$

Next subsection

Example 3: Usage of derivatives

Step 1 Given recurrence

$$g_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ \frac{2}{n} g_{n-2}, & \text{if } n > 0; \end{cases}$$

can be represented by the single equation

$$g_n = \frac{2}{n} g_{n-2} + [n = 0].$$

Some values:

n	0	1	2	3	4	5	6	7	8	9	10
g_n	1	0	1	0	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{1}{24}$	0	$\frac{1}{120}$

Example 3: Usage of derivatives (cont.)

Step 2 Write down $G(z) = \sum_n g_n z^n$ and its first derivative:

$$\begin{aligned} G(z) &= \sum_n g_n z^n \\ &= \sum_n [n=0]\, z^n + 2\sum_n \frac{g_{n-2}}{n} z^n \\ &= 1 + 2\sum_n \frac{g_{n-2}}{n} z^n \end{aligned}$$

Differentiating, we get a differential equation:

$$G'(z) = 2\sum_n \frac{g_{n-2}\cdot n}{n} z^{n-1} = 2z\sum_n g_{n-2} z^{n-2} = 2zG(z)$$

Example 3: Usage of derivatives (2)

Step 3 We need to solve the differential equation $G'(z) = 2zG(z)$.

- We rewrite the equation as

$$\frac{dG(z)}{dz} = 2zG(z)$$

- This is a separable differential equation, which can be solved by treating $G(z)$ as a variable:

$$\frac{dG(z)}{G(z)} = 2z\,dz$$

- By equating the indefinite integrals, we get:

$$\log G(z) = z^2 + C$$

  whence $G(z) = Ke^{z^2}$ where $K = e^C$.
- By applying $G(0) = g_0 = 1$ we get $K = 1$. Thus, $G(z) = e^{z^2}$.

Example 3: Usage of derivatives (3)

Step 4 Considering that $e^z = \sum_{n \geqslant 0} \frac{1}{n!} z^n$,

- and denoting $u = z^2$, we get

$$
\begin{aligned}
G(z) = e^{z^2} = e^u = \sum \frac{1}{n!} u^n \\
= \sum \frac{1}{n!} (z^2)^n = \sum \frac{1}{n!} z^{2n} \\
= \sum_n \frac{1}{\lfloor n/2 \rfloor !} [n \text{ is even}] z^n
\end{aligned}
$$

- To conclude:

$$
g_n = \begin{cases} \frac{1}{k!}, & \text{if } n = 2k, k \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}
$$

Q.E.D.

Example 3: Usage of derivatives (3)

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- and denoting $u = z^2$, we get

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Q.E.D.

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Next section

Convolutions

- Given two sequences:

$$\langle f_0, f_1, f_2, \ldots \rangle = \langle f_n \rangle \text{ and } \langle g_0, g_1, g_2, \ldots \rangle = \langle g_n \rangle$$

The convolution of $\langle f_n \rangle$ and $\langle g_n \rangle$ is the sequence

$$\langle f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 g_1 + f_2 g_0, \ldots \rangle = \left\langle \sum_k f_k g_{n-k} \right\rangle = \left\langle \sum_{k+\ell=n} f_k g_\ell \right\rangle.$$

- If $F(z)$ and $G(z)$ are generating functions on the sequences $\langle f_n \rangle$ and $\langle g_n \rangle$, then their convolution has the generating function $F(z) \cdot G(z)$.
- Three or more sequences can be convolved analogously, for example:

$$\langle f_n \rangle \langle g_n \rangle \langle h_n \rangle = \left\langle \sum_{j+k+\ell=n} f_j g_k h_\ell \right\rangle$$

and the generating function of this three-fold convolution is the product $F(z) \cdot G(z) \cdot H(z)$.

Convolutions

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Next subsection

Fibonacci convolution

To compute $\sum_k f_k f_{n-k}$ use Fibonacci generating function (in the form given by Theorem 1 and considering that $\sum_n (n+1)z^n = \frac{1}{(1-z)^2}$):

$$
\begin{aligned}
F^2(z) &= \left(\frac{1}{\sqrt{5}} \left(\frac{1}{1-\Phi z} - \frac{1}{1-\widehat{\Phi} z} \right) \right)^2 \\
&= \frac{1}{5} \left(\frac{1}{(1-\Phi z)^2} - \frac{2}{(1-\Phi z)(1-\widehat{\Phi} z)} + \frac{1}{(1-\widehat{\Phi} z)^2} \right) \\
&= \frac{1}{5} \sum_{n \geqslant 0} (n+1)\Phi^n z^n - \frac{2}{5} \sum_{n \geqslant 0} f_{n+1} z^n + \frac{1}{5} \sum_{n \geqslant 0} (n+1)\widehat{\Phi}^n z^n \\
&= \frac{1}{5} \sum_{n \geqslant 0} (n+1)(\Phi^n + \widehat{\Phi}^n) z^n - \frac{2}{5} \sum_{n \geqslant 0} f_{n+1} z^n \\
&= \frac{1}{5} \sum_{n \geqslant 0} (n+1)(2f_{n+1} - f_n) z^n - \frac{2}{5} \sum_{n \geqslant 0} f_{n+1} z^n \\
&= \frac{1}{5} \sum_{n \geqslant 0} (2nf_{n+1} - (n+1)f_n) z^n
\end{aligned}
$$

Hence:

$$\sum_k f_k f_{n-k} = \frac{2nf_{n+1} - (n+1)f_n}{5}.$$

Fibonacci convolution (2)

On the previous slide the following was used:

Property

For any $n \geqslant 0$: $\Phi^n + \widehat{\Phi}^n = 2f_{n+1} - f_n$

Proof

The equalities $\sum_n \Phi^n z^n = \frac{1}{1-\Phi z}$, $\sum_n \widehat{\Phi}^n z^n = \frac{1}{1-\widehat{\Phi} z}$, and $\Phi + \widehat{\Phi} = 1$ are used in the following derivation:

$$
\begin{aligned}
\sum_n (\Phi^n + \widehat{\Phi}^n) z^n &= \frac{1}{1-\Phi z} + \frac{1}{1-\widehat{\Phi} z} = \frac{1-\widehat{\Phi} z + 1 - \Phi z}{(1-\Phi z)(1-\widehat{\Phi} z)} = \\
&= \frac{2-z}{1-z-z^2} = \frac{2}{z} \cdot \frac{z}{1-z-z^2} - \frac{z}{1-z-z^2} = \\
&= \frac{2}{z} \sum_n f_n z^n - \sum_n f_n z^n = 2 \sum_n f_n z^{n-1} - \sum_n f_n z^n = \\
&= 2 \sum_n f_{n+1} z^n - \sum_n f_n z^n = \\
&= \sum_n (2f_{n+1} - f_n) z^n
\end{aligned}
$$

Q.E.D.

Fibonacci convolution (2)

On the previous slide the following was used:

Property

For any $n \geqslant 0$: $\Phi^n + \widehat{\Phi}^n = 2f_{n+1} - f_n$

Proof (alternative)

We know from Exercise 6.28 that

$$\Phi^n + \widehat{\Phi}^n = L_n = f_{n+1} + f_{n-1}\,,$$

with the convention $f_{-1} = 1$, is the nth Lucas number, which is the solution to the recurrence:

$$\begin{array}{ll} L_0 = 2\,; & L_1 = 1\,; \\ L_n = L_{n-1} + L_{n-2} & \forall n \geqslant 2\,. \end{array}$$

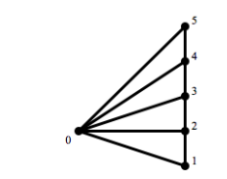
By writing the recurrence relation for Fibonacci numbers in the form $f_{n-1} = f_{n+1} - f_n$ (which, incidentally, yields $f_{-1} = 1$), we get precisely $L_n = 2f_{n+1} - f_n$.

Q.E.D.

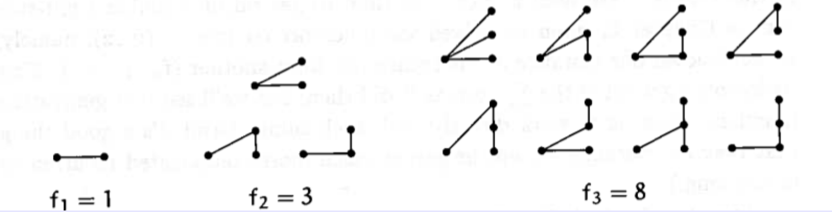
Next subsection

Spanning trees for fan

Example: the fan of order 5:



Spanning trees:



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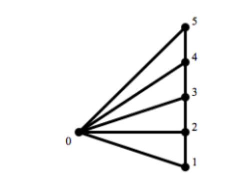

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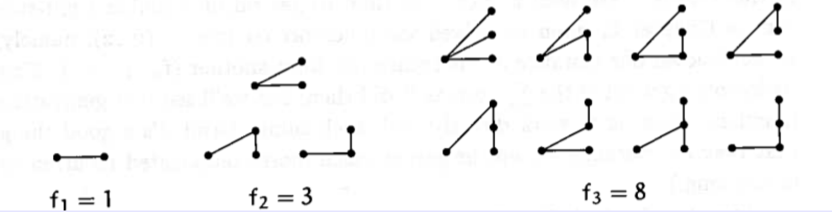
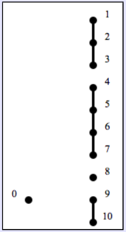


Spanning trees for fan (2)



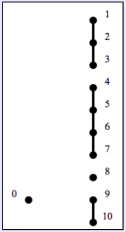
How many spanning trees can we make?

- We need to connect 0 to each of the four blocks:
  - two ways to join 0 with $\{9,10\}$,
  - one way to join 0 with $\{8\}$,
  - four ways to join 0 with $\{4,5,6,7,\}$,
  - three ways to join 0 with $\{1,2,3\}$,
- There is altogether $2 \cdot 1 \cdot 4 \cdot 3 = 24$ ways for that.

In general:

$$s_n = \sum_{m>0} \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,k_2,\ldots,k_m>0}} k_1 k_2 \cdots k_m$$

Spanning trees for fan (2)



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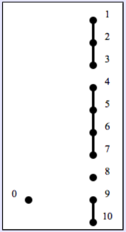
In general:

$$s_n = \sum_{m>0} \sum_{\substack{k_1 + k_2 + \cdots + k_m = n \\ k_1, k_2, \ldots, k_m > 0}} k_1 k_2 \cdots k_m$$

For example

$$f_4 = 4 + \underbrace{3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3}_{} + \underbrace{2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2}_{} + 1 \cdot 1 \cdot 1 \cdot 1 = 21$$

Spanning trees for fan (2)



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In general:

$$s_n = \sum_{m>0} \sum_{\substack{k_1 + k_2 + \cdots + k_m = n \\ k_1, k_2, \ldots, k_m > 0}} k_1 k_2 \cdots k_m$$

This is the sum of m-fold convolutions of the sequence $\langle 0, 1, 2, 3, \ldots \rangle$.

Spanning trees for fan (3)

Generating function for the number of spanning trees:

- The sequence $\langle 0, 1, 2, 3, \ldots \rangle$ has the generating function

$$G(z) = \frac{z}{(1-z)^2}.$$

- Hence the generating function for $\langle f_n \rangle$ is

$$\begin{aligned} S(z) = G(z) + G^2(z) + G^3(z) + \cdots &= \frac{G(z)}{1 - G(z)} \\ &= \frac{z}{(1-z)^2\left(1 - \frac{z}{(1-z)^2}\right)} \\ &= \frac{z}{(1-z)^2 - z} \\ &= \frac{z}{1 - 3z + z^2}. \end{aligned}$$

Spanning trees for fan (3)

Generating function for the number of spanning trees:

- The sequence $\langle 0, 1, 2, 3, \ldots \rangle$ has the generating function

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$$\begin{aligned} S(z) = G(z) + G^2(z) + G^3(z) + \cdots &= \frac{G(z)}{1 - G(z)} \\ &= \frac{z}{(1-z)^2\left(1 - \frac{z}{(1-z)^2}\right)} \\ &= \frac{z}{(1-z)^2 - z} \\ &= \frac{z}{1 - 3z + z^2}. \end{aligned}$$

Consequently $s_n = f_{2n}$

Next subsection

Dyck language

Definition

The Dyck[1] language is the language consisting of balanced strings of parentheses '[' and ']'.

Another definition

If $X = \{x, \overline{x}\}$ is the alphabet, then the Dyck language is the subset $\mathscr{D}$ of words u of X^* which satisfy

1. $|u|_x = |u|_{\overline{x}}$, where $|u|_x$ is the number of letters x in the word u, and
2. if u is factored as vw, where v and w are words of X^*, then $|v|_x \geqslant |v|_{\overline{x}}$.

	0 pairs	1 pair	2 pairs	3 pairs
Elements	∅	[ ]	[[ ]] [ ] [ ]	[[[ ]]] [[ ]][ ] [ ][[ ]] [[][]] [ ][ ][ ]
	∅	AB	AABB ABAB	AAABBB AABBAB ABAABB AABABB ABABAB
No of words	1	1	2	5

[1] Pronounced “Dück”. Walther von Dyck (1856-1934) was a German mathematician.

Dyck language (2)

- Let C_n be the number of words in the Dyck language $\mathscr{D}$ having exactly n pairs of parentheses.
- If $u = vw$ for $u \in \mathscr{D}$, then $v \in \mathscr{D}$ if and only if $w \in \mathscr{D}$.
- Then every word $u \in \mathscr{D}$ of length $\geqslant 2$ has a unique writing $u = [v]\,w$ such that $v, w \in \mathscr{D}$ (possibly empty) but $[p \notin \mathscr{D}$ for every prefix p of v (including v itself).
- Hence, for every $n > 0$,

$$C_n = C_0\,C_{n-1} + C_1\,C_{n-2} + \cdots + C_{n-1}\,C_0\,.$$

- The numbers C_n are called Catalan numbers, from the Belgian mathematician Eugène Catalan.
  Let us derive the closed formula for C_n in the following slides.

Catalan numbers

Step 1 The recurrence equation of Catalan numbers for all integers

$$C_n = \sum_k C_k C_{n-\mathbf{1}-k} + [n=0].$$

Step 2 Write down $C(z) = \sum_n C_n z^n$:

$$\begin{aligned}
C(z) = \sum_n C_n z^n &= \sum_{k,n} C_k C_{n-\mathbf{1}-k} z^n + \sum_n [n=0] z^n \\
&= \left(\sum_k C_k z^k\right) \cdot z \cdot \left(\sum_n C_{n-\mathbf{1}-k} z^{n-\mathbf{1}-k}\right) + 1 \\
&= \left(\sum_k C_k z^k\right) \cdot z \cdot \left(\sum_n C_n z^n\right) + 1 \\
&= z(C(z))^2 + 1
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Catalan numbers (2)

Step 3 Solving the quadratic equation $z(C(z))^2 - C(z) + 1 = 0$ for C(z) yields:

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

(The solution with "+" isn't proper as it must be $\lim_{z\to 0} C(z) = 1$.)

Step 4 From the binomial theorem we get:

$$\sqrt{1-4z} = \sum_{k\geqslant 0} \binom{1/2}{k} (-4z)^k = 1 + \sum_{k\geqslant 1} \frac{1}{2k} \binom{-1/2}{k-1} (-4z)^k$$

- Using the equality for binomials $\binom{-1/2}{n} = (-1/4)^n \binom{2n}{n}$ we finally get

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{k\geqslant 1} \frac{1}{k} \binom{-1/2}{k-1} (-4z)^{k-1}$$

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Proof that $\binom{-1/2}{n} = (-1/4)^n \binom{2n}{n}$

We prove a bit more: for every $r \in \mathbb{R}$ and $k \geqslant 0$,

$$r^{\underline{k}} \cdot \left(r - \frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}$$

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Then for $r = k = n$, dividing by $(n!)^2$ and using $n^{\underline{n}} = n!$,

$$\binom{n-1/2}{n} = \left(\frac{1}{4}\right)^n \binom{2n}{n} \; ;$$

and as $r^{\underline{k}} = (-1)^k (-r)^{\overline{k}} = (-1)^k (-r+k-1)^{\underline{k}}$, for $k = n$ and $r = n - 1/2$ we get:

$$\binom{-1/2}{n} = \binom{-(n-1/2)+n-1}{n} = \frac{(-1)^n}{4^n} \binom{2n}{n}$$

Q.E.D.

Resume Catalan numbers

Formulas for computation

- $C_{n+1} = \frac{2(2n+1)}{n+2} C_n$, with $C_0 = 1$
- $C_n = \frac{1}{n+1}\binom{2n}{n}$
- $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \binom{2n-1}{n} - \binom{2n-1}{n+1}$
- Generating function: $C(z) = \frac{1-\sqrt{1-4z}}{2z}$

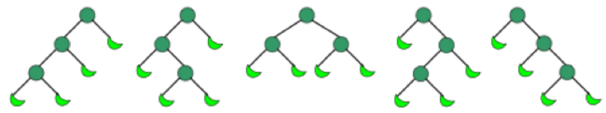
Eugéne Charles Catalan
(1814–1894)

$$\lim_{n\to\infty} \frac{C_n}{C_{n-1}} = 4$$

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1 430	4 862	16 796

Applications of Catalan numbers

Number of complete binary trees with $n+1$ leaves is C_n



The Dyck language consists of exactly n characters A and n characters B, and every prefix does not contain more B-s than A-s. For example, there are five words with 6 letters in the Dyck language:

AAABBB AABABB AABBAB ABAABB ABABAB

Corollary

C_n is the number of words of length $2n$ in the Dyck language.

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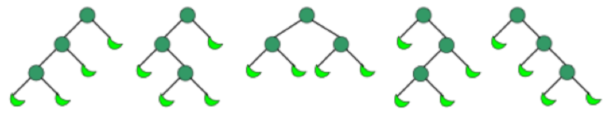

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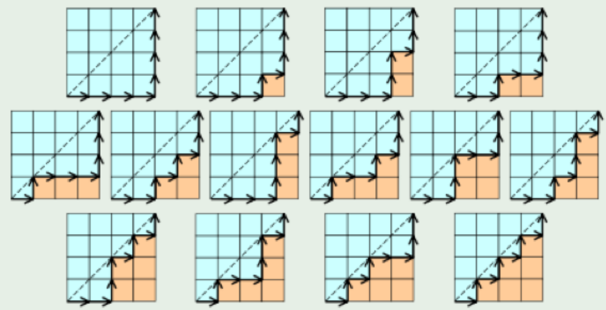
Applications of Catalan numbers (2)

Monotonic paths

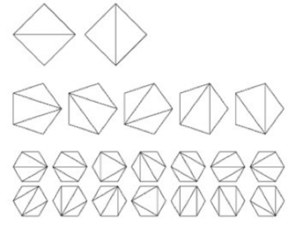
C_n is the number of monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards.



Applications of Catalan numbers (3)

Polygon triangulation

C_n is the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.



See more applications, for example, on
http://www.absoluteastronomy.com/topics/Catalan_number

Next section

Exponential generating function

Definition

The exponential generating function (briefly, egf) of the sequence $\langle g_n \rangle$ is the function

$$\widehat{G}(z) = \sum_{n \geqslant 0} \frac{g_n}{n!} z^n ,$$

that is, the generating function of the sequence $\left\langle \frac{g_n}{n!} \right\rangle$.

For example, $e^z = \sum_{n \geqslant 0} \frac{z^n}{n!}$ is the egf of the constant sequence 1.

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Why exponential generating functions?

- Because $\langle g_n / n! \rangle$ might have a "simpler" generating function than $\langle g_n \rangle$.
- Because if $\limsup_{n \geqslant 0} \sqrt[n]{|g_n|} = +\infty$ then $\langle g_n \rangle$ does not have a generating function analytic in a neighborhood of the origin. Example: the Bernoulli numbers.
- Because $\langle g_n \rangle$ might count labeled objects so that there are $n!$ labels for every object of size n, and $\langle g_n \rangle$ gives the same information as $\langle g_n / n! \rangle$.

Exponential generating functions: Basic maneuvers

Let $\widehat{F}(z)$ and $\widehat{G}(z)$ be the exponential generating functions of $\langle f_n \rangle$ and $\langle g_n \rangle$.

As usual, we put $f_n = g_n = 0$ for every $n < 0$, and $\text{undefined} \cdot 0 = 0$.

- $\alpha \widehat{F}(z) + \beta \widehat{G}(z) = \sum_n \left(\frac{\alpha f_n + \beta g_n}{n!} \right) z^n$
- $\widehat{G}(cz) = \sum_n \frac{c^n g_n}{n!} z^n$
- $z \widehat{G}(z) = \sum_n \frac{n g_{n-1}}{n!} z^n$
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TAL TECH

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TAL TECH

Binomial convolution

Definition

The binomial convolution of the sequences $\langle f_n \rangle$ and $\langle g_n \rangle$ is the sequence $\langle h_n \rangle$ defined by:

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Examples

- $\langle (a+b)^n \rangle$ is the binomial convolution of $\langle a^n \rangle$ and $\langle b^n \rangle$.
- If $\widehat{F}(z)$ is the egf of $\langle f_n \rangle$ and $\widehat{G}(z)$ is the egf of $\langle g_n \rangle$, then $\widehat{H}(z) = \widehat{F}(z) \cdot \widehat{G}(z)$ is the egf of $\langle h_n \rangle$, because then:

$$\frac{h_n}{n!} = \sum_k \frac{f_k}{k!} \frac{g_{n-k}}{(n-k)!}\,, \text{ which is equivalent to } h_n = \sum_k \frac{n!}{k!(n-k)!} f_k g_{n-k}\,.$$

Bernoulli numbers and exponential generating functions

Recall that the Bernoulli numbers are defined by the recurrence:

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = [m=0] \ \ \forall m \geqslant 0 \,,$$

which is equivalent to:

$$\sum_{n} \binom{n}{k} B_k = B_n + [n=1] \ \ \forall n \geqslant 0 \,.$$

The left-hand side is a binomial convolution with the constant sequence 1. Then the egf $\widehat{B}(z)$ of the Bernoulli numbers satisfies

$$\widehat{B}(z) \cdot e^z = \widehat{B}(z) + z \ :$$

which yields

$$\widehat{B}(z) = \frac{z}{e^z - 1} \,.$$

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To make a comparison:

$$\sum_{n \geqslant 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1} \text{ but } \sum_{n \geqslant 0} B_n^+ z^n = \frac{1}{z} \frac{d^2}{dz^2} \ln \int_0^\infty t^{z-1} e^{-t} dt$$

where $B_n^+ = B_n \cdot [B_n \geqslant 0]$.