Asymptotics

ITT9132 Concrete Mathematics
Lecture 16 – 13 May 2019

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Another application of power series

Theorem

Let $G(z) = \sum_{k \geqslant 0} g_k z^k$ have convergence radius $R > 0$. If either:

- $f(n) \prec 1$, or
- $R = \infty$ and $f(n) = O(1)$,

then for every $m \geqslant 1$, if $h(n) = O(f(n))$ then:

$$G(h(n)) = \sum_{0 \leqslant k < m} g_k (h(n))^k + O((f(n))^m)$$

Remark:

- $f(n) \prec 1$ if and only if $\lim_{n \to \infty} f(n) = 0$: that is, f vanishes at infinity.
- $f(n) = O(1)$ if and only if $|f(n)| \leqslant C$ for some $C > 0$ and every n: that is, f is bounded.
- The thesis is equivalent to saying that:

$$\sum_{k \geqslant m} g_k (h(n))^k = O((f(n))^m) \text{ whenever } h(n) = O(f(n)).$$

TAL TECH

Another application of power series

Theorem

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Examples:

- $\ln(1 + O(1/n)) = O(1/n)$.
- $\ln(1 + 1/n) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + O\left(\frac{1}{n^4}\right)$.
- $e^{\frac{\ln n}{n}} = 1 + \frac{\ln n}{n} + \frac{(\ln n)^2}{2n^2} + O\left(\left(\frac{\ln n}{n}\right)^3\right)$.
- $e^{1-1/n} = \frac{5}{2} - \frac{2}{n} + \frac{1}{2n^2} + O\left(\left(1 - \frac{1}{n}\right)^3\right)$.

  This holds because $e^z = \sum_{k \geqslant 0} \frac{z^k}{k!}$ has infinite convergence radius.

Another application of power series

Theorem

Let $G(z) = \sum_{k \geqslant 0} g_k z^k$ have convergence radius $R > 0$. If either:

- $f(n) \prec 1$, or
- $R = \infty$ and $f(n) = O(1)$,

then for every $m \geqslant 1$, if $h(n) = O(f(n))$ then:

$$G(h(n)) = \sum_{0 \leqslant k < m} g_k (h(n))^k + O((f(n))^m)$$

Proof for $R < \infty$ and $f \prec 1$:

- Let $h(n) = O(f(n))$. Choose $C > 0$ such that $|h(n)| \leqslant C \cdot |f(n)|$ for every $n \geqslant 0$.
- Fix $\delta \in (0, R)$. Then $K_m = C^m \cdot \sum_{k \geqslant m} |g_k| \delta^{k-m} < \infty$.
- Choose n_0 such that $|f(n)| < \delta / C$ for every $n \geqslant n_0$: for such n, $|h(n)| < \delta$.
- Then for $n \geqslant n_0$:

$$\left| \sum_{k \geqslant m} g_k (h(n))^k \right| < |h(n)|^m \cdot \sum_{k \geqslant m} |g_k| \cdot \delta^{k-m} \leqslant K_m \cdot |f(n)|^m .$$

Another application of power series

Theorem

Let $G(z) = \sum_{k \geqslant 0} g_k z^k$ have convergence radius $R > 0$. If either:

- $f(n) \prec 1$, or
- $R = \infty$ and $f(n) = O(1)$,

then for every $m \geqslant 1$, if $h(n) = O(f(n))$ then:

$$G(h(n)) = \sum_{0 \leqslant k < m} g_k (h(n))^k + O((f(n))^m)$$

Proof for $R = \infty$ and $f(n) = O(1)$:

- Let $h(n) = O(f(n))$. Choose $C > 0$ such that $|h(n)| \leqslant C \cdot |f(n)|$ for every $n \geqslant 0$.
- Fix $\delta > 0$ such that $|f(n)| \leqslant \delta / C$ for every n. Then $|h(n)| \leqslant \delta$ for every n too.
- As $R = \infty$, $K_m = C^m \cdot \sum_{k \geqslant m} |g_k| \delta^{k-m} < \infty$.
- Then for every n:

$$\left| \sum_{k \geqslant m} g_k (h(n))^k \right| \leqslant |h(n)|^m \cdot \sum_{k \geqslant m} |g_k| \cdot \delta^{k-m} \leqslant K_m \cdot |f(n)|^m .$$

Next section

Next subsection

Bootstrapping

To get an asymptotics for the solution of a recurrence:

1. first, take a rough estimate;
2. next, substitute the estimate into the recurrence.

This is an example of "pulling oneself up by one's bootstraps".

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k \geqslant 1} \frac{z^k}{k^2}}$.

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Step 1: Look for a simple asymptotics

As we have an exponential, it might be a good idea to differentiate by series, and see if a more manageable formula appears:

$$\sum_{n\geqslant 1} n g_n z^{n-1} = G(z) \cdot \sum_{k\geqslant 1} \frac{z^{k-1}}{k} = \sum_{n\geqslant 1} \sum_{0\leqslant k<n} \frac{g_k}{n-k} z^{n-1}$$

which gives us the recurrence:

$$n g_n = \sum_{k=0}^{n-1} \frac{g_k}{n-k} \quad \forall n \geqslant 1 .$$

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k \geqslant 1} \frac{z^k}{k^2}}$.

Step 1: Look for a simple asymptotics

A numerical computation of the first values gives:

n	0	1	2	3	4	5	6
g_n	1	1	$\frac{3}{4}$	$\frac{19}{36}$	$\frac{107}{288}$	$\frac{641}{2400}$	$\frac{51103}{259200}$

It looks like all the terms are positive and not larger than 1: which can easily be proved by induction. Then: $g_n = O(1)$.

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k\geqslant 1} \frac{z^k}{k^2}}$.

Step 2: Plug in the asymptotics

If we replace the g_k on the right-hand side with $O(1)$, we find:

$$ng_n = \sum_{k=0}^{n-1} \frac{O(1)}{n-k} = \sum_{k=1}^{n} \frac{O(1)}{k} = H_n \cdot O(1) = O(\log n).$$

Then: $g_n = O\left(\frac{\log n}{n}\right)$.

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k \geqslant 1} \frac{z^k}{k^2}}$.

Step 2: Plug in the asymptotics *again*

If we replace the g_k on the right-hand side with $O(\log(k)/k)$ and put the first summand out of the sum, we find:

$$ng_n = \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{n-k} O\left(\frac{\log k}{k}\right) = \frac{1}{n} + \sum_{k=1}^{n-1} \frac{O(\log k)}{k(n-k)} = \frac{1}{n} + \frac{2}{n} H_{n-1} O(\log n),$$

because $\frac{1}{k(n-k)} = \frac{1}{n}\left(\frac{1}{k} + \frac{1}{n-k}\right)$ and $k = O(n)$. Then $g_n = O\left(\left(\frac{\log n}{n}\right)^2\right)$.

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k\geqslant 1} \frac{z^k}{k^2}}$.

Step 3: Manipulate the estimate obtained

We now rewrite our original recurrence by "pulling out the largest part":

$$ng_n = \sum_{k=0}^{n-1} \frac{g_k}{n} + \sum_{k=0}^{n-1} g_k \left(\frac{1}{n-k} - \frac{1}{n} \right) = \frac{1}{n} \sum_{k\geqslant 0} g_k - \frac{1}{n} \sum_{k\geqslant n} g_k + \frac{1}{n} \sum_{k=0}^{n-1} \frac{kg_k}{n-k}.$$

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k\geqslant 1} \frac{z^k}{k^2}}$.

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As $\sum_{k\geqslant 1} \dfrac{z^k}{k^2}$ converges at $z=1$, the first sum is "simply" $G(1) = e^{\sum_{k\geqslant 1} 1/k^2} = e^{\pi^2/6}$.
For the second sum, we observe that:

$$\sum_{k>n} \frac{\log^2 k}{k^2} < \sum_{m\geqslant 1} \sum_{n^m<k\leqslant n^{m+1}} \frac{\log^2 n^{m+1}}{k(k-1)} < \sum_{m\geqslant 1} \frac{(m+1)^2 \log^2 n}{n^m} = O\left(\frac{\log^2 n}{n}\right),$$

so $\sum_{k\geqslant n} g_k = \sum_{k\geqslant n} O\left(\dfrac{\log^2 k}{k^2}\right) = O\left(\sum_{k\geqslant n} \dfrac{\log^2 k}{k^2}\right) = O\left(O\left(\dfrac{\log^2 n}{n}\right)\right) = O\left(\dfrac{\log^2 n}{n}\right)$.

Example: Estimating a growth rate

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Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k\geqslant 1} \frac{z^k}{k^2}}$.

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We now rewrite our original recurrence by "pulling out the largest part":

$$ng_n = \sum_{k=0}^{n-1} \frac{g_k}{n} + \sum_{k=0}^{n-1} g_k \left(\frac{1}{n-k} - \frac{1}{n}\right) = \frac{1}{n}\sum_{k\geqslant 0} g_k - \frac{1}{n}\sum_{k\geqslant n} g_k + \frac{1}{n}\sum_{k=0}^{n-1} \frac{kg_k}{n-k}.$$

For the third and last sum, we again plug in the asymptotics:

$$\begin{aligned}
\sum_{0\leqslant k<n} \frac{kg_k}{n-k} &= \sum_{0<k<n} \frac{k}{n-k} O\left(\left(\frac{\log k}{k}\right)^2\right) = O\left(\sum_{0<k<n} \frac{\log^2 k}{k(n-k)}\right) \\
&= O\left(\sum_{0<k<n} \frac{1}{n}\left(\frac{1}{k} + \frac{1}{n-k}\right)\log^2 n\right) \\
&= O\left(H_{n-1}\frac{\log^2 n}{n}\right) = O\left(\frac{\log^3 n}{n}\right)
\end{aligned}$$

We can take out the big-O because all summands are nonnegative.

Example: Estimating a growth rate

Problem

Find an asymptotic estimate for $g_n = [z^n]G(z)$ where $G(z) = e^{\sum_{k\geqslant 1} \frac{z^k}{k^2}}$.

Step 3: Manipulate the estimate obtained

We now rewrite our original recurrence by "pulling out the largest part":

$$ng_n = \sum_{k=0}^{n-1} \frac{g_k}{n} + \sum_{k=0}^{n-1} g_k \left(\frac{1}{n-k} - \frac{1}{n} \right) = \frac{1}{n} \sum_{k\geqslant 0} g_k - \frac{1}{n} \sum_{k\geqslant n} g_k + \frac{1}{n} \sum_{k=0}^{n-1} \frac{kg_k}{n-k}.$$

We conclude:

$$\begin{aligned} g_n &= \frac{e^{\pi^2/6}}{n^2} + O\left(\frac{1}{n^2} O\left(\frac{\log^2 n}{n} \right) \right) + O\left(\frac{1}{n^2} O\left(\frac{\log^3 n}{n} \right) \right) \\ &= \frac{e^{\pi^2/6}}{n^2} + O\left(\left(\frac{\log n}{n} \right)^3 \right). \end{aligned}$$

Next subsection

Trading tails: The trick

Suppose we must approximate a function of the form:

$$f(n) = \sum_k a_k(n).$$

1. For each n, split the ks between a dominant set D_n and a tail set T_n.
2. Find asymptotic estimates $a_k(n) = b_k(n) + O(c_k(n))$ which hold for $k \in D_n$.
3. Make sure that the following three sums are all "small":

$$S_a(n) = \sum_{k \in T_n} a_k(n)\,; \;\; S_b(n) = \sum_{k \in T_n} b_k(n)\,; \;\; S_c(n) = \sum_{k \in D_n} |c_k(n)|\,.$$

Then the following estimate holds:

$$f(n) = \sum_k b_k(n) + O(S_a(n)) + O(S_b(n)) + O(S_c(n)).$$

Trading tails: The rationale

Suppose we performed the three steps in the previous slide. Then:

- For k in the dominant set, we have the estimate:

$$\begin{aligned}\sum_{k\in D_n} a_k(n) &= \sum_{k\in D_n} (b_k(n) + O(c_k(n))) \\ &= \left(\sum_{k\in D_n} b_k(n)\right) + O(S_c(n))\end{aligned}$$

  Note that, to pass the big-O outside the summation, we needed the absolute values of the $c_k(n)$.

- For k in the tail set, we have the estimate:

$$\begin{aligned}\sum_{k\in T_n} a_k(n) &= \sum_{k\in T_n} (b_k(n) + a_k(n) - b_k(n)) \\ &= \left(\sum_{k\in T_n} b_k(n)\right) + O(S_a(n)) + O(S_b(n))\end{aligned}$$

  We did not need the absolute values of $a_k(n)$ and $b_k(n)$, because they did appear explicitly in the summation, not inside a big-O.

TAL TECH

Trading tails: Example

Find an asymptotic estimate for the following sequence:

$$L_n = \sum_{k \geqslant 0} \frac{\ln(n+2^k)}{k!}$$

We immediately observe that, if k is "small" and n is "large", then:

$$\ln(n+2^k) = \ln\left(n \cdot \left(1 + \frac{2^k}{n}\right)\right) = \ln n + \frac{2^k}{n} - \frac{2^{2k}}{2n^2} + O\left(\frac{2^{3k}}{n^3}\right)$$

and we can surely use this approximation within the convergence radius of the Taylor series of $\ln(1+z)$ at the origin of the complex plane.
Such convergence radius is 1, so we require $2^k < n$, that is, $k < \lfloor \lg n \rfloor$.

Trading tails: Example

Find an asymptotic estimate for the following sequence:

$$L_n = \sum_{k \geqslant 0} \frac{\ln(n+2^k)}{k!}$$

Step 1 For every $n \geqslant 1$ we set:

$$D_n = [0 : \lfloor \lg n \rfloor - 1] \text{ and } T_n = \mathbb{N} \setminus D_n = \{\lfloor \lg n \rfloor, \lfloor \lg n \rfloor + 1, \ldots\}$$

Step 2 For $k \in D_n$ we write $a_k(n) = b_k(n) + O(c_k(n))$ where:

$$\begin{aligned} a_k(n) &= \frac{\ln(n+2^k)}{k!}; \\ b_k(n) &= \frac{1}{k!}\left(\ln n + \frac{2^k}{n} - \frac{4^k}{2n^2}\right); \\ c_k(n) &= \frac{8^k}{n^3 \cdot k!}. \end{aligned}$$

Trading tails: Example

Find an asymptotic estimate for the following sequence:

$$L_n = \sum_{k \geqslant 0} \frac{\ln(n + 2^k)}{k!}$$

Step 3c The estimate for $S_c(n)$ is immediate:

$$\sum_{0 \leqslant k < \lfloor \lg n \rfloor} \frac{8^k}{n^3 \cdot k!} \leqslant \frac{1}{n^3} \cdot \sum_{k \geqslant 0} \frac{8^k}{k!} = \frac{e^8}{n^3} = O\left(\frac{1}{n^3}\right).$$

Trading tails: Example

Find an asymptotic estimate for the following sequence:

$$L_n = \sum_{k \geqslant 0} \frac{\ln(n+2^k)}{k!}$$

Step 3b To estimate $S_b(n)$ we can replace $\dfrac{2^k}{n} - \dfrac{4^k}{2n^2}$ with $2^k + 4^k$ and find:

$$\left| \sum_{k \geqslant \lfloor \lg n \rfloor} b_k(n) \right| < \sum_{k \geqslant \lfloor \lg n \rfloor} \frac{\ln n + 2^k + 4^k}{k!}$$

$$< \frac{\ln n + 2^{\lfloor \lg n \rfloor} + 4^{\lfloor \lg n \rfloor}}{\lfloor \lg n \rfloor!} \cdot \sum_{k \geqslant 0} \frac{4^k}{k!} .$$

The last sum is e^4, while the fraction is $O\left(\frac{n^2}{\lfloor \lg n \rfloor!}\right)$ because of the summand $4^{\lfloor \lg n \rfloor} = n^2$. But $\lfloor \lg n \rfloor!$ grows faster than any power of n, so $S_b(n)$ is *really very small* for large n.

Trading tails: Example

Find an asymptotic estimate for the following sequence:

$$L_n = \sum_{k \geqslant 0} \frac{\ln(n+2^k)}{k!}$$

Step 3a The estimate for $S_a(n)$ is easy to compute: for $n \geqslant 2$ and $k \geqslant 1$,

$$\sum_{k \geqslant \lfloor \lg n \rfloor} \frac{\ln(n+2^k)}{k!} \leqslant \sum_{k \geqslant \lfloor \lg n \rfloor} \frac{\ln(n \cdot 2^k)}{k!} < \sum_{k \geqslant \lfloor \lg n \rfloor} \frac{k + \ln n}{k!} = O\left(\frac{1}{\lfloor \lg n \rfloor !}\right)$$

because $\ln 2^k < \ln e^k = k$.

Trading tails: Example

Find an asymptotic estimate for the following sequence:

$$L_n = \sum_{k \geqslant 0} \frac{\ln(n+2^k)}{k!}$$

We can now summarize:

$$\begin{aligned} L_n &= \sum_{k \geqslant 0} \frac{1}{k!} \left(\ln n + \frac{2^k}{n} - \frac{4^k}{2n^2} \right) + O\left(\frac{1}{\lfloor \lg n \rfloor !} \right) + O\left(\frac{n^2}{\lfloor \lg n \rfloor !} \right) + O\left(\frac{1}{n^3} \right) \\ &= e \ln n + \frac{e^2}{n} - \frac{e^4}{2n^2} + O\left(\frac{1}{n^3} \right). \end{aligned}$$

Next section

Euler's summation formula

Theorem

If $f(x)$ is differentiable m times in an open interval which contains $[a:b]$, then:

$$\sum_{a \leqslant k < b} f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!} \left(f^{(k-1)}(b) - f^{(k-1)}(a) \right) + R_m \,,$$

where B_k is the kth Bernoulli number and where:

$$R_m = (-1)^{m+1} \int_a^b \frac{B_m(\{x\})}{m!} f^{(m)}(x)\,dx \,,$$

where, in turn, $B_m(x) = \sum_k \binom{m}{k} B_k x^{m-k}$ is the mth Bernoulli polynomial.

Euler's summation formula: A special case

If $f(x) = x^{m-1}$ with m positive integer, then Euler's summation formula becomes:

$$\begin{aligned}
\sum_{a \leqslant k < b} x^{m-1} &= \frac{b^m - a^m}{m} + \sum_{k=1}^{m} \frac{B_k}{k!} (m-1)^{\underline{k-1}} (b^{m-k} - a^{m-k}) + 0 \\
&= \frac{1}{m} \sum_{k=0}^{m} B_k \frac{m(m-1)^{\underline{k-1}}}{k!} (b^{m-k} - a^{m-k}) \quad \text{because } B_0 = 1 \\
&= \frac{1}{m} \sum_{k=0}^{m} \binom{m}{k} (b^{m-k} - a^{m-k})
\end{aligned}$$

Euler's summation formula: The rationale

Let $\sum$, Δ, $\int$ and D be the operators of summation, difference, integration, and differentiation, respectively.

- Suppose that f is smooth: that is, it has derivatives of any order.
- Taylor's formula tells us that: $f(x+\varepsilon) = f(x) + f'(x)\,\varepsilon + \frac{f''(x)}{2}\,\varepsilon^2 + \dots$
- For $\varepsilon = 1$, and writing $D^k f$ in place of $f^{(k)}$, this becomes:

$$\Delta f(x) = \sum_{k \geqslant 1} \frac{D^k f(x)}{k!} = (e^D - 1) f(x),$$

  with what looks like a little abuse of notation ...
- Now, if $\Delta = e^D - 1$, then its inverse $\sum$ must be:

$$\sum = \frac{1}{e^D - 1} = \frac{1}{D} \cdot \frac{D}{e^D - 1} = \frac{1}{D} \cdot \left(1 + \sum_{k \geqslant 1} \frac{B_k}{k!} D^k \right) = \int + \sum_{k \geqslant 1} \frac{B_k}{k!} D^{k-1},$$

  which is Euler's summation formula with an infinite sum and no remainder.

Euler's summation formula: the proof

We give the proof by induction on $m \geqslant 1$ with $a = 0$ and $b = 1$.
We only need this case, because if $a \leqslant c \leqslant b$, then:

- $\sum_{a \leqslant k < b} f(k) = \sum_{a \leqslant k < c} f(k) + \sum_{c \leqslant k < b} f(k)$;
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, and similar for $\int_a^b \frac{B_m(\{x\})}{m!} f^{(m)}(x)\,dx$;
- $f^{(k-1)}(b) - f^{(k-1)}(a) = (f^{(k-1)}(c) - f^{(k-1)}(a)) + (f^{(k-1)}(b) - f^{(k-1)}(c))$ for every k from 1 to m; and
- if $a \neq 0$ we can replace $f(x)$ with $g(x) = f(x+a)$.

So what we need to prove is:

$$f(0) = \int_0^1 f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!} \left(f^{(k-1)}(1) - f^{(k-1)}(0) \right) - (-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\,dx$$

for every function f differentiable m times in (s, t) for some $s < 0$ and $t > 1$.

Euler's summation formula: the proof

We give the proof by induction on $m \geqslant 1$ with $a = 0$ and $b = 1$.

Base case: $m = 1$. Then $B_1 = -1/2$, $B_1(x) = x - 1/2$, and the formula becomes:

$$f(0) = \int_0^1 f(x)\,dx - \frac{1}{2}(f(1) - f(0)) + \int_0^1 \left(x - \frac{1}{2}\right) f'(x)\,dx\,,$$

or equivalently,

$$\frac{f(0) + f(1)}{2} = \int_0^1 f(x)\,dx + \int_0^1 \left(x - \frac{1}{2}\right) f'(x)\,dx\,,$$

But the right-hand side is precisely:

$$\begin{aligned}
\int_0^1 \left(f(x) + xf'(x)\right)\,dx - \frac{1}{2}\int_0^1 f'(x)\,dx &= xf(x)|_0^1 - \frac{1}{2}(f(1) - f(0)) \\
&= f(1) - \frac{1}{2}f(1) + \frac{1}{2}f(0) \\
&= \frac{f(0) + f(1)}{2}\,.
\end{aligned}$$

Q.E.D.

Euler's summation formula: the proof

We give the proof by induction on $m \geqslant 1$ with $a = 0$ and $b = 1$.

Induction: Suppose the thesis is true for $m - 1 \geqslant 1$. Proving it for m is equivalent to proving that $R_m = R_{m-1} - \frac{B_m}{m!}(f^{(m-1)}(1) - f^{(m-1)}(0))$, that is:

$$
\begin{aligned}
(-1)^m B_m \left(f^{(m-1)}(1) - f^{(m-1)}(0) \right) \quad = \quad & m \int_0^1 B_{m-1}(x) f^{(m-1)}(x)\, dx \\
& + \int_0^1 B_m(x) f^{(m)}(x)\, dx
\end{aligned}
$$

Euler's summation formula: the proof

We give the proof by induction on $m \geqslant 1$ with $a=0$ and $b=1$.

Induction: Suppose the thesis is true for $m-1 \geqslant 1$. Proving it for m is equivalent to proving that $R_m = R_{m-1} - \frac{B_m}{m!}(f^{(m-1)}(1) - f^{(m-1)}(0))$, that is:

$$\begin{aligned}(-1)^m B_m \left(f^{(m-1)}(1) - f^{(m-1)}(0)\right) &= m\int_0^1 B_{m-1}(x) f^{(m-1)}(x)\,dx \\ &\quad + \int_0^1 B_m(x) f^{(m)}(x)\,dx\end{aligned}$$

Now, if $B_m(x) = \sum_k \binom{m}{k} B_k x^{m-k}$ (which is the case) then:

$$\begin{aligned}\frac{d}{dx}B_m(x) &= \sum_k \binom{m}{k}(m-k)B_k x^{m-1-k} \\ &= \sum_k \frac{m^{\underline{k}}(m-k)}{k!} B_k x^{m-1-k} \\ &= \sum_k \frac{m(m-1)^{\underline{k}}}{k!} B_k x^{m-1-k} \\ &= m\sum_k \binom{m-1}{k} B_k x^{m-1-k} = mB_{m-1}(x).\end{aligned}$$

Euler's summation formula: the proof

We give the proof by induction on $m \geqslant 1$ with $a = 0$ and $b = 1$.

Induction: Suppose the thesis is true for $m - 1 \geqslant 1$. Proving it for m is equivalent to proving that $R_m = R_{m-1} - \frac{B_m}{m!}(f^{(m-1)}(1) - f^{(m-1)}(0))$, that is:

$$\begin{aligned}(-1)^m B_m \left(f^{(m-1)}(1) - f^{(m-1)}(0)\right) &= m \int_0^1 B_{m-1}(x) f^{(m-1)}(x)\,dx \\ &\quad + \int_0^1 B_m(x) f^{(m)}(x)\,dx\end{aligned}$$

But since $\frac{d}{dx} B_m(x) = m B_{m-1}(x)$, the right-hand side is:

$$\begin{aligned}&\int_0^1 \left(\left(\frac{d}{dx} B_m(x)\right) f^{(m-1)}(x) + B_m(x) \frac{d}{dx} f^{(m-1)}(x)\right) dx \\ = &\int_0^1 \left(\frac{d}{dx} B_m(x) f^{(m-1)}(x)\right) dx \\ = &\ B_m(1) f^{(m-1)}(1) - B_m(0) f^{(m-1)}(0)\end{aligned}$$

TAL TECH

Euler's summation formula: the proof

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$$(-1)^m B_m \left(f^{(m-1)}(1) - f^{(m-1)}(0) \right) = B_m(1) f^{(m-1)}(1) - B_m(0) f^{(m-1)}(0)$$

But the above can be rewritten:

$$g(1) f^{(m-1)}(1) - g(0) f^{(m-1)}(0) = 0 \text{ with } g(x) = B_m(x) - (-1)^m B_m$$

and this must hold for every f differentiable m times: the only possibility is that $g(0) = g(1) = 0$, that is,

$$B_m(0) = B_m(1) = (-1)^m B_m$$

and this must hold whatever $m \geqslant 2$ is.

Euler's summation formula: the proof

We give the proof by induction on $m \geqslant 1$ with $a = 0$ and $b = 1$.

Induction: Suppose the thesis is true for $m - 1 \geqslant 1$. Proving it for m is equivalent to proving that

$$B_m(0) = B_m(1) = (-1)^m B_m\,,\ \ m \geqslant 2\,.$$

But $B_m(0) = B_m(1)$ for $m \geqslant 2$ follows directly from the defining equation of Bernoulli numbers:

$$\sum_k \binom{m}{k} B_k = B_m + [m = 1] \ \text{for every } m \geqslant 1$$

and the $(-1)^m$ sign is not a problem, because for odd $m > 1$ it is $B_m = 0$. Q.E.D.

TAL TECH

Euler's summation formula and asymptotics: Idea

As $B'_m(x) = mB_{m-1}(x)$ for every $m \geqslant 0$, from our discussion follows that:

$$\int_0^1 B_m(x)\,dx = \frac{B_{m+1}(1) - B_{m+1}(0)}{(m+1)!} = 0 \quad \text{for every } m \geqslant 1\,.$$

Then the remainder R_m is the integral of the product of an mth derivative with a function of average zero, everything divided by a factorial:

$$R_m = \frac{(-1)^{m+1}}{m!} \int_a^b B_m(\{x\}) f^{(m)}(x)\,dx = \frac{(-1)^{m+1}}{m!} \sum_{a \leqslant k < b} \int_0^1 B_m(x) f^{(m)}(x+k)\,dx$$

Such a quantity has good chances to be small, even if the B_m grow very large.
Actually, as $\sum_{m \geqslant 0} \frac{B_m}{m!} z^m = \frac{z}{e^z - 1}$ and the right-hand side is differentiable in the entire complex plane, the left-hand side has infinite convergence radius, so $B_m/m!$ vanishes faster than exponentially.

Behavior of $B_m(x)$ for $x \in [0,1]$

We have observed that $B'_m(x) = mB_{m-1}(x)$ for every $m \geqslant 1$. If $x \in [0,1]$ then:

- $B_1(x) = x - 1/2$ is negative in $(0,1/2)$ and positive in $(1/2,1)$, and $B_1(1-x) = -B_1(x)$.
- Then $B_2(x)$ is decreasing in $(0,1/2)$ and increasing in $(1/2,1)$, and by comparing derivatives, $B_2(1-x) = B_2(x)$. Also, $B_2(0) = B_2 = 1/6 > 0$.
- By comparing derivatives, $B_3(1-x) = -B_3(x)$. As $B_3(0) = B_3 = 0$ and $B'_3(x) = 3B_2(x) > 0$ near 0, $B_3(x)$ is positive in $(0,1/2)$ and negative in $(1/2,1)$.
- Then $B_4(x)$ is increasing in $(0,1/2)$ and decreasing in $(1/2,1)$, and by comparing derivatives, $B_4(1-x) = B_4(x)$. Also, $B_4(0) = B_4 = -1/30 < 0$.
- By comparing derivatives, $B_5(1-x) = -B_5(x)$. As $B_5(0) = B_5 = 0$ and $B'_5(x) = 5B_4(x) < 0$ near 0, $B_5(x)$ is negative in $(0,1/2)$ and positive in $(1/2,1)$.
- And so on ...

Behavior of $B_m(x)$ at $x = 1/2$

From the previous slide we deduce that for $x \in [0,1]$, $|B_{2m}(x)|$ is maximum at either $x = 0$ or $x = 1/2$.

Lemma

For every $m \geqslant 0$, $B_m(1/2) = \left(2^{1-m} - 1\right) B_m$.

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Proof $B_m(x) = \sum_k \binom{m}{k} B_k x^{m-k}$ is the mth term of the binomial convolution of $\langle B_m \rangle$ and $\langle x^m \rangle$. Then:

$$\sum_{m \geqslant 0} \frac{B_m(x)}{m!} z^m = \frac{z e^{xz}}{e^z - 1},$$

which for $x = 1/2$ becomes:

$$\begin{aligned}
\sum_{m \geqslant 0} \frac{B_m(1/2)}{m!} z^m &= \frac{z e^{z/2}}{e^z - 1} \\
&= 2 \frac{(z/2)(e^{z/2} + 1)}{e^z - 1} - \frac{z}{e^z - 1} \\
&= 2 \sum_{m \geqslant 0} \frac{B_m}{m!} \left(\frac{z}{2}\right)^m - \sum_{m \geqslant 0} \frac{B_m}{m!} z^m \\
&= \sum_{m \geqslant 0} \frac{(2^{1-m} - 1) B_m}{m!} z^m
\end{aligned}$$

Behavior of $B_m(x)$ at $x = 1/2$

From the previous slide we deduce that for $x \in [0,1]$, $|B_{2m}(x)|$ is maximum at either $x = 0$ or $x = 1/2$.

Lemma

For every $m \geqslant 0$, $B_m(1/2) = \left(2^{1-m} - 1\right) B_m$.

As $|2^{1-m} - 1| < 1$, we conclude:

Corollary

For every $x \in [0,1]$ and integer $m \geqslant 1$, $|B_{2m}(x)| \leqslant |B_{2m}| = (-1)^{m-1} B_{2m}$.

Euler's summation formula and asymptotics: Estimates

Let us write Euler's summation formula again:

$$\sum_{a\leqslant k<b} f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right) + \frac{(-1)^{m+1}}{m!}\int_a^b B_m(\{x\})f^{(m)}(x)\,dx\,.$$

Since $B_m = 0$ for $m > 1$ odd and $B_1 = -\frac{1}{2}$, we can only consider m even and rewrite:

$$\begin{aligned}\sum_{a\leqslant k<b} f(k) \quad &= \quad \int_a^b f(x)\,dx - \frac{f(b)-f(a)}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) \\ &\quad - \frac{1}{(2m)!}\int_a^b B_{2m}(\{x\})f^{(2m)}(x)\,dx\,.\end{aligned}$$

But for $x \in [0,1]$ it is $|B_{2m}(x)| \leqslant |B_{2m}|$, and as $\dfrac{(2\pi)^{2m}}{2}\dfrac{|B_{2m}|}{(2m)!} = \sum_{k\geqslant 1}\dfrac{1}{k^{2m}}$ (cf. Ch. 6)

$$\begin{aligned}\sum_{a\leqslant k<b} f(k) \quad &= \quad \int_a^b f(x)\,dx - \frac{f(b)-f(a)}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) \\ &\quad + O\left((2\pi)^{-2m}\right)\int_a^b |f^{(2m)}(x)|\,dx\,.\end{aligned}$$

Euler's summation formula and asymptotics: More estimates

If $f^{(2m)}$ is nonnegative in $[a,b]$, then:

$$|R_{2m}| \leqslant \frac{|B_{2m}|}{(2m)!} \int_a^b f^{(2m)}(x)\,dx = \frac{|B_{2m}|}{(2m)!} \left(f^{(2m-1)}(b) - f^{(2m-1)}(a) \right)$$

But as $B_{2m+1} = 0$ for $m \geqslant 1$, it is $R_{2m} = R_{2m+1}$, so the first discarded term when we approximate to the $2m$th order instead of the $(2m+2)$nd must be $R_{2m} - R_{2m+2}$.

Lemma

If $f^{(2m+2)}(x) \geqslant 0$ for every $x \in [a,b]$, then $(-1)^m R_{2m} \geqslant 0$.

Proof for $a = 0$ and $b = 1$: (general case follows easily)

- $R_{2m} = R_{2m+1} = \frac{1}{(2m+1)!} \int_0^1 B_{2m+1}(x) f^{(2m+1)}(x)\,dx$.
- As $f^{(2m+2)} \geqslant 0$, $f^{(2m+1)}$ is nondecreasing, and since B_{2m+1} is symmetric around $x = 1/2$, the second half of the sinusoid counts more than the first.
- For m even, B_{2m+1} is negative in $(0, 1/2)$ and positive in $(1/2, 1)$; for m odd, B_{2m+1} is positive in $(0, 1/2)$ and negative in $(1/2, 1)$. The thesis follows.

Next section

Next subsection

A bell-shaped approximation

Problem

Give an asymptotic approximation of:

$$\Theta_n = \sum_k e^{-k^2/n}$$

A bell-shaped approximation

Problem

Give an asymptotic approximation of:

$$\Theta_n = \sum_k e^{-k^2/n}$$

Let us think first of the behavior of the summand:

- The function $f(x) = e^{-x^2}$ is maximum at $x = 0$ with $f(0) = 1$; stays near 1 in $[-1,1]$; and becomes very small very quickly for $x \to \pm\infty$.
- Then $f_n(x) = e^{-x^2/n}$ stays near 1 for $|x| \leqslant \sqrt{n}$, and vanishes quickly for $x \to \pm\infty$.
- We then expect Θ_n to be of the form $\Theta_n \approx C \cdot \sqrt{n}$.

Euler's summation formula at infinity

Since Euler's summation formula holds for every $a \leqslant b$, it also holds for the limits for $a \to -\infty$ and $b \to +\infty$ when they exist.
This is the case for $f_n(x) = e^{-x^2/n}$, so:

$$\begin{aligned}\sum_k e^{-k^2/n} &= \int_{-\infty}^{+\infty} e^{-x^2/n}\,dx \\ &+ \sum_{k=1}^{m} \frac{B_k}{k!}\left(\lim_{x\to+\infty} f_n^{(k-1)}(x) - \lim_{x\to-\infty} f_n^{(k-1)}(x)\right) \\ &+(-1)^{m+1}\int_{-\infty}^{+\infty} \frac{B_m(\{x\})}{m!} f_n^{(m)}(x)\,dx\,.\end{aligned}$$

Now, for $f(x) = e^{-x^2}$ it is $f^{(k)}(x) = P_k(x)e^{-x^2}$ for a polynomial $P_m(x)$ of degree k.
As $f_n(x) = f(x/\sqrt{n})$, we have $f_n^{(k)}(x) = n^{-k/2} f^{(k)}(x/\sqrt{n}) \to 0$ for $x \to \pm\infty$, hence:

$$\sum_k e^{-k^2/n} = \sqrt{\pi n} + (-1)^{m+1}\int_{-\infty}^{+\infty} \frac{B_m(\{x\})}{m!} f_n^{(m)}(x)\,dx \text{ for suitable } C > 0\,.$$

The Gaussian integral

Theorem

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}\,.$$

The Gaussian integral

Theorem

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.$$

Proof:

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} e^{-x^2}\, dx\right)^2 &= \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy \\
&= \int_{\theta=0}^{2\pi} \int_{\rho=0}^{+\infty} e^{-\rho^2} \rho\, d\rho\, d\theta \\
&= 2\pi \cdot \frac{1}{2} \int_0^{+\infty} e^{-t}\, dt \\
&\quad \text{with the change of variable } t = \rho^2 \\
&= \pi \cdot \left[-e^{-t}\right]_0^{+\infty} = \pi\,.
\end{aligned}$$

Q.E.D.

The Gaussian integral

Theorem

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.$$

Corollary

$$\int_{-\infty}^{+\infty} e^{-x^2/n}\, dx = \sqrt{\pi n}\,.$$

An estimate for the error

The absolute error we make by approximating Θ_n with $\sqrt{\pi n}$ is:

$$\begin{aligned}
(-1)^{m+1}\int_{-\infty}^{+\infty}\frac{B_m(\{x\})}{m!}f_n^{(m)}(x)\,dx &= \frac{(-1)^{m+1}}{n^{m/2}}\int_{-\infty}^{+\infty}\frac{B_m(\{x\})}{m!}f^{(m)}\left(\frac{x}{\sqrt{n}}\right)dx \\
&= \frac{(-1)^{m+1}}{n^{(m-1)/2}}\int_{-\infty}^{+\infty}\frac{B_m(\{t\sqrt{n}\})}{m!}f^{(m)}(t)\,dt \\
&\quad \text{with the change of variable } t = x/\sqrt{n} \\
&= O\left(n^{(1-m)/2}\right)
\end{aligned}$$

because $B_m(x)$ is bounded in $[0,1]$ and $\int_{-\infty}^{+\infty}|f^{(m)}(x)|\,dx$ is finite.
But $m \geqslant 1$ is arbitrary, because $f(x)$ is smooth in $\mathbb{R}$: we conclude

$$\Theta_n = \sqrt{\pi n} + O\left(n^{-M}\right) \text{ for any } M > 0\,.$$

Next subsection

Stirling's approximation

Theorem

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \ln \sqrt{2\pi} + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)$$

Stirling's approximation

Theorem

$$\ln n! = \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)$$

Proof: (sketch; see the textbook for details)

1. Prove that there exists a constant σ such that:

$$\ln n! = \left(n+\frac{1}{2}\right)\ln n - n + \sigma + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right).$$

2. Use the formula from point 1, the trading tails technique, and the approximation for Θ_n from the previous section to prove that:

$$\sum_k \binom{2n}{k} = 2^{2n}\frac{\sqrt{2\pi}}{e^\sigma}\left(1+O\left(n^{-1/2+3\varepsilon}\right)\right) \text{ for } 0<\varepsilon<\frac{1}{6}.$$

3. Conclude that it must be $\sigma = \ln\sqrt{2\pi}$.