ITT9132 – Concrete Mathematics
Midterm Test

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Full name: Code:

1. Take note of the code near your full name: it will be used to display the results.
2. Write your solution on the page of the corresponding exercise, explaining your reasoning.
3. You may use any formula seen in classroom or appearing in the self-evaluation tests.
4. You may use the additional paper to draft your answers. However, only what is written in the exercises' pages will be evaluated.
5. Partially completed exercises may receive a fraction of the total score.
6. Only handwritten notes are allowed.
7. Electronic devices, including mobile phones must be turned off. Using a pocket or tabletop calculator is allowed as the only exception.
8. It is forbidden to leave the room without having returned the assignment.

Exercise 1

(10 points) Solve the recurrence:

$$\begin{aligned} T_0 &= 1\,; \\ 3T_n &= 4T_{n-1} + \left(\frac{2}{3}\right)^n \quad \forall n \geqslant 1\,. \end{aligned}$$

Solution. The system has the form:

$$\begin{aligned} a_0 T_0 &= 1\,; \\ a_n T_n &= b_n T_{n-1} + c_n \quad \forall n \geqslant 1 \end{aligned}$$

with:

$$a_0 = 1\,; \quad a_n = 3 \text{ for every } n \geqslant 1\,; \quad b_n = 4\,; \quad c_n = \left(\frac{2}{3}\right)^n$$

This suggests using a summation factor s_n such that $s_n b_n = s_{n-1} a_{n-1}$ for every $n \geqslant 1$. By using the procedure from the book, we get:

$$\begin{aligned} s_0 &= 1 \text{ as usual;} \\ s_1 &= \frac{a_0}{b_1} = \frac{1}{4}\,; \\ s_n &= s_{n-1} \cdot \frac{a_{n-1}}{b_n} = \frac{3^{n-1}}{4^n} \quad \text{for every } n \geqslant 2\,. \end{aligned}$$

By multiplying by s_n and setting $U_n = s_n a_n T_n = \left(\frac{3}{4}\right)^n T_n$ the system becomes:

$$\begin{aligned} U_0 &= 1\,; \\ U_n &= U_{n-1} + \frac{3^{n-1}}{4^n} \left(\frac{2}{3}\right)^n \\ &= U_{n-1} + \frac{1}{3 \cdot 2^n} \forall n \geqslant 1 \end{aligned}$$

which clearly has the solution:

$$\begin{aligned} U_n &= 1 + \sum_{k=1}^{n} \frac{1}{3 \cdot 2^k} \\ &= 1 + \frac{1}{3} \cdot \left(1 - \frac{1}{2^n}\right) \\ &= \frac{4}{3} - \frac{1}{3 \cdot 2^n}\,. \end{aligned}$$

The solution of the original system is then:

$$T_n = \left(\frac{4}{3}\right)^n U_n = \left(\frac{4}{3}\right)^n \cdot \left(\frac{4}{3} - \frac{1}{3 \cdot 2^n}\right) .$$

For $n = 0$ the formula correctly returns $T_0 = 1$.

Exercise 2

(6 points) Prove that $n^{13} - n$ is divisible by 195 for every integer $n \geqslant 1$.

Solution. As $195 = 3 \cdot 5 \cdot 13$ as a product of (powers of) primes, we must prove that $n^{13} - n$ is divisible by 3, 5 and 13. For 13, because of Fermat's little theorem, there is no problem; for the other two factors, we must proceed with more caution.

Let us first deal with the prime factor 3. We observe that:

$$\begin{aligned} n^{13} - n &= n \cdot (n^{12} - 1) \\ &= n \cdot (n^2 - 1) \cdot (n^{10} + n^8 + n^6 + n^4 + n^2 + 1) \\ &= (n^3 - n) \cdot (n^{10} + n^8 + n^6 + n^4 + n^2 + 1)\,, \end{aligned}$$

and the factor $n^3 - n$ is a multiple of 3 because of Fermat's little theorem.

The argument for the prime factor 5 is similar, but we use a different decomposition:

$$n^{13} - n = n \cdot (n^4 - 1) \cdot (n^8 + n^4 + 1) = (n^5 - n) \cdot (n^8 + n^4 + 1)\,.$$

Exercise 3

(10 points) Find a closed form for

$$\sum_{1\leqslant k\leqslant n} k(k-1)\,2^{-k}$$

as a function of n, and use it to compute

$$\sum_{k\geqslant 1} k^{\underline{2}}\,2^{-k}$$

Hint: what is $\sum_{1\leqslant k\leqslant n} k\,2^{-k}$?

Solution. We can find a closed form by either the perturbation method, or discrete calculus.

- Perturbation method:

  Let $S_n = \sum_{1\leqslant k\leqslant n} k(k-1)\,2^{-k}$. Then the following chain of equalities holds:

$$\begin{aligned}
S_n + (n+1)n\,2^{-n-1} &= 0 + \sum_{k=2}^{n+1} k(k-1)\,2^{-k} \\
&= \sum_{k=1}^{n} (k+1)k\,2^{-k-1} \\
&= \frac{1}{2}\sum_{k=1}^{2} (k^2+k)\,2^{-k} \\
&= \frac{1}{2}\sum_{k=1}^{2} (k^2-k+2k)\,2^{-k} \\
&= \frac{1}{2}\left(S_n + 2\sum_{k=1}^{n} k\,2^{-k}\right) \\
&= \frac{S_n}{2} + 2\sum_{k=1}^{n} k\,2^{-k}\,.
\end{aligned}$$

  Multiplying both sides by 2, we get:

$$\begin{aligned}
2S_n + (n+1)n\,2^{-n} &= S_n + 2\sum_{k=1}^{n} k\,2^{-k} \\
&= S_n + 2\left(2 - (n+2)\,2^{-n}\right),
\end{aligned}$$

as we have seen during exercise session 9. From this we get:

$$\begin{aligned} S_n &= 4 - (2n+4)\,2^{-n} - (n+1)n\,2^{-n} \\ &= 4 - (2n+4+n^2+n)\,2^{-n} \\ &= 4 - (n^2+3n+4)\,2^{-n}\,. \end{aligned}$$

- Discrete calculus:

  We use summation by parts with $u(x) = x^{\underline{2}}$ and $\Delta v(x) = (1/2)^x$. Then $\Delta u(x) = 2x$ and $v(x) = -2 \cdot (1/2)^x$, as we have seen during exercise session 9. Then:

$$\begin{aligned} \sum x^{\underline{2}} \left(\frac{1}{2}\right)^x \delta x &= -2x^{\underline{2}} \left(\frac{1}{2}\right)^x + 4 \sum x \left(\frac{1}{2}\right)^{x+1} \\ &= -2x^{\underline{2}} \left(\frac{1}{2}\right)^x + 4 \sum x \left(\frac{1}{2}\right)^{x+1} \\ &= -2x^{\underline{2}} \left(\frac{1}{2}\right)^x + 2 \sum x \left(\frac{1}{2}\right)^x . \end{aligned}$$

  Then

$$\begin{aligned} \sum_1^{n+1} x^{\underline{2}} \left(\frac{1}{2}\right)^x \delta x &= -2\, x^{\underline{2}} \left(\frac{1}{2}\right)^x \Bigg|_1^{n+1} + 2 \sum_1^{n+1} x \left(\frac{1}{2}\right)^x \\ &= -2(n+1)n\,2^{-n-1} + 0 + 2 \cdot (2 - (n+2)\,2^{-n})\,, \end{aligned}$$

  as we have seen during exercise session 9. Reorganizing, we conclude:

$$\sum_{k=1}^{n} k^{\underline{2}}\, 2^{-k} = 4 - 2^{-n} \cdot ((n+1)n + 2(n+2)) = 4 - (n^3+3n+4)\,2^{-n}\,.$$

Note that for $n = 1$ we have $n^2+3n+4 = 8$, so the formula correctly returns $S_1 = 0$. By taking the limit for $n \to \infty$ we find:

$$\sum_{k \geqslant 1} k^{\underline{2}}\, 2^{-k} = 4\,.$$

Exercise 4

(4 points) Express

$$\sum_{k=1}^{n} \left\lceil \sqrt{\left\lfloor \sum_{j=0}^{k} \frac{1}{j!} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z}\right]} \right\rceil$$

as a function of n. *Hint:* this is a "don't panic" question and the answer is rather simple.

Solution. The scary part is the argument of the square root. The cubic root inside the Iverson brackets looks like a false alarm, because that summand will either be 0 or 1: let's focus on the floor instead. We know from Calculus that the series $\sum_{j \geqslant 0} 1/j!$ converges to $e = 2.71828\ldots < 3$: as for $k = 1$ it is $1/0! + 1/1! = 1 + 1 = 2$, the floor is always 2. Then the argument of the square root is always either 2 or 3, and its ceiling is always 2. In conclusion,

$$\sum_{k=1}^{n} \left\lceil \sqrt{\left\lfloor \left(1 + \frac{1}{k}\right)^{k} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z}\right]} \right\rceil = 2n\,.$$