ITT9132 – Concrete Mathematics
Midterm Test – Recovery

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Full name: Code:

1. Take note of the code near your full name: it will be used to display the results.
2. Write your solution on the page of the corresponding exercise, explaining your reasoning.
3. You may use any formula seen in classroom or appearing in the self-evaluation tests.
4. You may use the additional paper to draft your answers. However, only what is written in the exercises' pages will be evaluated.
5. Partially completed exercises may receive a fraction of the total score.
6. Only handwritten notes are allowed.
7. Electronic devices, including mobile phones must be turned off. Using a pocket or tabletop calculator is allowed as the only exception.
8. It is forbidden to leave the room without having returned the assignment.

Exercise 1 (10 points)

Solve the recurrence:

$$\begin{aligned} T_0 &= 2\,; \\ 2T_n &= 5T_{n-1} + \frac{5^n}{n \cdot 2^{n-1}} \quad \forall n \geqslant 1\,. \end{aligned}$$

Solution. The form of the recurrence suggests using a summation factor. If we rewrite:

$$\begin{aligned} T_0 &= 2\,; \\ a_n T_n &= b_n T_{n-1} + c_n \quad \forall n \geqslant 1 \end{aligned}$$

we see that it is:

$$a_0 = 1\,; \quad a_n = 2 \; \forall n \geqslant 1\,; \quad b_n = 5\,; \quad c_n = \frac{5^n}{n \cdot 2^{n-1}}$$

A summation factor s_n such that $s_n b_n = s_{n-1} a_{n-1}$ is then computed as follows:

$$s_0 = 1\,; \quad s_1 = \frac{a_0}{b_1} = \frac{1}{5}\,; \quad s_n = s_{n-1} \frac{a_{n-1}}{b_n} = \frac{2^{n-1}}{5^n} \; \forall n \geqslant 2\,.$$

Then the original recurrence is equivalent to:

$$\begin{aligned} T_0 &= 2\,; \\ s_n a_n T_n &= s_n b_n T_{n-1} + s_n c_n \\ &= s_{n-1} a_{n-1} T_{n-1} + s_n c_n \quad \forall n \geqslant 1 \end{aligned}$$

which, by putting $U_n = s_n a_n T_n = \left(\frac{2}{5}\right)^n$, becomes:

$$\begin{aligned} U_0 &= 2\,; \\ U_n &= U_{n-1} + \frac{1}{n} \quad \forall n \geqslant 1 \end{aligned}$$

which clearly has the solution:

$$U_n = 2 + \sum_{k=1}^{n} \frac{1}{n} = 2 + H_n$$

The solution of our original recurrence is thus:

$$T_n = \left(\frac{5}{2}\right)^n U_n = \left(\frac{5}{2}\right)^n (2 + H_n)\,.$$

Exercise 2 (6 points)

Prove that $n^{11} - n^7 - n^5 + n$ is divisible by 70 for every integer $n \geqslant 0$.

Solution. We have $70 = 2 \cdot 5 \cdot 7$ so $n^{11} - n^7 - n^5 + n$ is divisible by 70 if and only if it is divisible by 2, 5 and 7. But:

$$\begin{aligned} n^{11} - n^7 - n^5 + n &= n(n^{10} - n^6 - n^4 + 1) \\ &= n(n^6(n^4 - 1) - (n^4 - 1)) \\ &= n(n^4 - 1)(n^6 - 1) \\ &= n(n-1)(n+1)(n^2+1)(n^6-1)\,, \end{aligned}$$

which is divisible by 2, 5 and 7 because of Fermat's little theorem.

Exercise 3 (10 points)

Express

$$\sum_{k=1}^{n-1} k^{\underline{-3}} H_k$$

as a function of n, and use it to compute

$$\sum_{k \geqslant 1} k^{\underline{-3}} H_k \, .$$

Hint: $\lim_{n\to\infty} H_n/n = 0$.

Solution. As $\sum_{k=1}^{n-1} k^{\underline{-3}} H_k = \sum_1^n x^{\underline{-3}} H_x \,\delta x$, it seems natural to use discrete calculus and summation by parts. Let $u(x) = H_x$ and $\Delta v(x) = x^{\underline{-3}}$, so that $\Delta u(x) = x^{\underline{-1}}$ and $v(x) = -\frac{1}{2} x^{\underline{-2}}$: then,

$$\begin{aligned}
\sum_1^n x^{\underline{-3}} H_x \,\delta x &= \sum_1^n u(x)\,\Delta v(x)\,\delta x \\
&= u(x)v(x)\big|_1^n - \sum_1^n Ev(x)\Delta u(x)\,\delta x \\
&= -\frac{1}{2} x^{\underline{-2}} H_x \bigg|_1^n + \frac{1}{2}\sum_1^n (x+1)^{\underline{-2}} x^{\underline{1}}\,\delta x \\
&= -\frac{1}{2}\cdot\frac{1}{(n+1)(n+2)}\cdot H_n + \frac{1}{2}\cdot\frac{1}{2\cdot 3} + \frac{1}{2}\sum_1^n x^{\underline{-3}}\,\delta x \\
&= \frac{1}{12} - \frac{H_n}{1(n+1)(n+2)} - \frac{1}{4} x^{\underline{-2}}\bigg|_1^n \\
&= \frac{1}{12} - \frac{H_n}{1(n+1)(n+2)} - \frac{1}{4(n+1)(n+2)} + \frac{1}{4\cdot 2\cdot 3} \\
&= \left(\frac{1}{2} + \frac{1}{24}\right) - \frac{1}{2(n+1)(n+2)}\left(H_n + \frac{1}{2}\right) \\
&= \frac{1}{8} - \frac{2H_n + 1}{4(n+1)(n+2)}
\end{aligned}$$

The hint says that the second summand on the last line vanishes for $n \to \infty$.

Then:

$$\begin{aligned}\sum_{k\geqslant 1} k^{\underline{-3}} H_k &= \lim_{n\to\infty} \sum_{k=1}^{n-1} k^{\underline{-3}} H_k \\ &= \lim_{n\to\infty} \left(\frac{1}{8} - \frac{2H_n + 1}{4(n+1)(n+2)} \right) \\ &= \frac{1}{8}\,.\end{aligned}$$

Exercise 4 (4 points total)

Let x be a real number and k a nonnegative integer. Prove that:

1. (2 points) If $x > 0$ then $\lfloor kx \rfloor \geqslant k \lfloor x \rfloor$.
2. (2 points) If $x < 0$ then $\lceil kx \rceil \leqslant k \lceil x \rceil$. *Hint:* Use point 1.

Solution. First, let $x > 0$. Write $x = \lfloor x \rfloor + \{x\}$: then,

$$\begin{array}{rcl} \lfloor kx \rfloor & = & \lfloor k \lfloor x \rfloor + k \{x\} \rfloor \\ & = & k \lfloor x \rfloor + \lfloor k \{x\} \rfloor \\ & & \text{because } k \lfloor x \rfloor \text{ is an integer} \\ & \geqslant & k \lfloor x \rfloor \ , \end{array}$$

because the second summand is nonnegative by the choice of k and the definition of $\{x\}$.

Now, suppose $x < 0$. Then:

$$\begin{array}{rcl} \lceil kx \rceil & = & - \lfloor -kx \rfloor \\ & = & - \lfloor k \cdot (-x) \rfloor \\ & \leqslant & -k \lfloor -x \rfloor \\ & & \text{by point 1 and because of the minus sign} \\ & = & k \lceil x \rceil \ . \end{array}$$

As a last minute note: there is actually no need of the hypothesis on the sign of x, because $\{x\}$ is nonnegative.