ITT9132 Concrete Mathematics
Introductory exercises

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Inclusion-exclusion formula

How many integer numbers between 1 and 1000 are divisible by either 7 or 11, but not both?

Solution. This exercise takes a twist on the classical *inclusion-exclusion formula* for the probabilities of two events:

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)\,.$$

The twist is that we are considering a *disjoint* union, so we have to remove $P(A \cap B)$ *twice* from $P(A) + P(B)$.[1]

As $1000 = 142 * 7 + 6 = 90 * 11 + 10 = 12 * 77 + 76$, there are 142 numbers between 1 and 1000 that are divisible by 7, 90 that are divisible by 11, and 12 that are divisible by 77. By the observation above, there are $142+90-12\cdot 2 = 208$ integer numbers between 1 and 1000 divisible by either 7 or 11, but not both.

The recurrence equation for binomial coefficients

Recall that, for integers $n \geq 0$ and $0 \leq k \leq n$, the *binomial coefficient* $\binom{n}{k}$, read "n choose k", is the number of ways we can choose k objects of a set with n elements, without taking into account the order in which we take them. Then:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}\,, \tag{1}$$

[1] Thanks to Ahto Truu for this remark.

where $k!$ is the number of ways to order a set of k elements. Note that $0! = \binom{n}{0} = \binom{n}{n} = 1$ for every $n \geq 0$, and $k! = k \cdot (k-1)!$ for every $k \geq 1$.

Prove that, for every $n \geq 0$ and every $1 \leq k \leq n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{2}$$

Solution (By calculation)**.** As $0 < k < n+1$, the following derivation is valid:

$$\begin{aligned}
\binom{n+1}{k} &= \frac{(n+1) \cdot n \cdots (n+1-k+1)}{k!} \\
&= \frac{n+1}{k} \cdot \frac{n \cdots (n-(k-1)+1)}{(k-1)!} \\
&= \left(\frac{n+1-k}{k} + 1 \right) \cdot \binom{n}{k-1} \\
&= \frac{n-k+1}{k} \cdot \frac{n \cdots (n-(k-1)+1)}{(k-1)!} + \binom{n}{k-1} \\
&= \binom{n}{k} + \binom{n}{k-1}.
\end{aligned}$$

Solution (By interpretation)**.** The left-hand side of (2) is the number of ways we can choose k objects from a set with $n+1$ elements, without taking the order into account. To do so, we have to decide whether we choose the first element, or not.

No. If we don't choose the first element, then all the k objects must be chosen between the other n elements: which can be done in $\binom{n}{k}$ ways.

Yes. If we choose the first element, then the remaining $k-1$ objects will have to be chosen between the remaining n objects: which can be done in $\binom{n}{k-1}$ ways.

Then the total of ways is precisely the right-hand side of (2).

Children and sweets

How many ways are there to distribute n identical sweets between k children (with $1 \leq k \leq n$) if each child must receive at least one sweet?

Solution (By experiment, intuition, and induction)**.** We make a first attempt with $n = 5$ and several values of k:

$k = 1$. Clearly, there is only one way of distributing the five sweets to the unique child!

$k = 2$. The first child will take either 1, 2, 3, or 4 sweets, and the other one will get those that are left: so there are 4 ways of distributing 5 sweets between 2 children, each child receiving at least one sweet.

$k = 3$. There are two possibilities: either one kid is *favored* in that s/he gets three sweets and the other two only get one each, or one kid is *disfavored* in the sense that s/he gets only one candy, and the other two get two each. In each case there are three ways to choose the (dis)favored child, so overall there are $3 + 3 = 6$ ways of distributing 5 sweets between 3 children, each child receiving at least one sweet.

$k = 4$. In this case, one of the children will receive two sweets, and the others will receive one each: there are thus 4 ways of distributing 5 sweets between 4 children, each child receiving at least one sweet.

$k = 5$. Clearly, there is only one way to distribute the five sweets between five children, who will have one each.

We then plot the values of the number $D(n, k)$ of the ways of distributing n sweets between k children (with $1 \leq k \leq n$) giving at least one sweet to each child, for $n = 5$ and $1 \leq k \leq 5$, and we observe a coincidence:

k	1	2	3	4	5
$D(5, k)$	1	4	6	4	1
$\binom{4}{k-1}$	1	4	6	4	1

Maybe it is true in general that $D(n, k) = \binom{n-1}{k-1}$? Let's try to prove it by induction *on* n: that is, we construct the proposition

$$P(n) \ : \ D(n, k) = \binom{n-1}{k-1} \ \forall k \in \{1, \ldots, n\}$$

and we prove, as the *induction base*, that $P(1)$ is true; and as the *inductive step*, that for every $n \geq 1$, if $P(n)$ is true, then so is $P(n + 1)$.

- *Induction base:* there is only $1 = \binom{0}{0}$ way of giving one sweet to one child, so that the child has at least one sweet. Therefore, $P(1)$ is true.

- *Inductive step:* Suppose $P(n)$ is true: that is, for every $k \in \{1, \dots, n\}$, there exist exactly $\binom{n-1}{k-1}$ ways of distributing n sweets between k children, giving at least one sweet to each child. What if the sweets are $n + 1$? Again, if the children are either 1 or $n + 1$, there is only $1 = \binom{n}{0} = \binom{n}{n}$ way: all to one, or one each, respectively. So we only need to consider the case $k \in \{2, \dots, n\}$.

  One idea to reduce the problem to a previous case, is to first distribute n sweets between the k children, then decide which child give the $(n+1)$st sweet. This idea, however, does not take into account that the sweets are identical, but the children are not!

  The issue is resolved if we observe that either the first child receives only one sweet, or he receives two or more. The last case corresponds to a division of n sweets between k children: as we are working under the hypothesis that $P(n)$ is true, we can assert that there are $\binom{n-1}{k-1}$ such distributions. The first case corresponds to a division of n sweets between $k - 1$ children: again, as we are working under the hypothesis that $P(n)$ is true, we can assert that there are $\binom{n-1}{k-2}$ such distributions. Adding the two together, for $k \in \{2, \dots, n\}$ there are

  $$\binom{n-1}{k-1} + \binom{n-1}{k-2} = \binom{n}{k-1}$$

  ways of distributing $n+1$ sweets between k children, so that each child receives at least one sweet. We have thus proved that, if $P(n)$ is true, then so is $P(n + 1)$: and that such implication holds for every $n \geq 1$, because n always acted only as a parameter, and no special cases needed to be treated.

Solution (By change of point of view)**.** Since the sweets are identical, we can distribute them by putting them in a line, and give the leftmost ones to the current child. Distributing n sweets between k children so that each child receives at least one sweet, is then the same of choosing at which $k - 1$ points we stop giving sweets to the current child, and go on to the next. There are $n - 1$ points where we can switch from a child to the next one, so by definition there must be $\binom{n-1}{k-1}$ ways of distributing n identical sweets between k children (with $1 \leq k \leq n$) giving each child at least one sweet.

Estimating sums

Consider the two sums:

$$A = \sum_{i=1}^{3} i^2 \sum_{j=1}^{i} (j^2+1) \ \ ; \ \ B = \sum_{j=1}^{3} (j^2+1) \sum_{i=j}^{3} i^2 \, .$$

Which one is larger?

Solution. The two sums are equal. To see this more easily, we introduce the *Iverson brackets* as a function from the set $\{\text{True}, \text{False}\}$ to the set $\{0,1\}$ defined as follows:

1. $[\text{True}] = 1$ and $[\text{False}] = 0$.

2. If a is either infinite or undefined, then $a \cdot [\text{False}] = 0$.

The Iverson brackets allow us to move the dependencies between the summation indices from the indices themselves to the summands. Then:

$$A = \sum_{i=1}^{3} \sum_{j=1}^{3} i^2(j^2+1)[j \leq i] \ \text{ and } \ B = \sum_{j=1}^{3} \sum_{i=1}^{3} (j^2+1) i^2 [i \geq j] \, ,$$

which are easily seen to be equal.

A more "brutal" alternative is to check which pairs (i,j) will enter each summation. For the first sum, the pairs are:

$$(1,1), (2,1), (2,2), (3,1), (3,2), (3,3) \, ;$$

for the second sum, they are:

$$(1,1), (2,1), (3,1), (2,2), (3,2), (3,3) \, .$$

So the pairs of indices are the same for both sums, and so are the summands corresponding to each pair: therefore, the sums must be equal. Now do it with 100 instead of 3 ...