ITT9132 Concrete Mathematics

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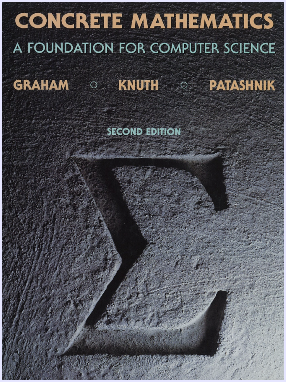
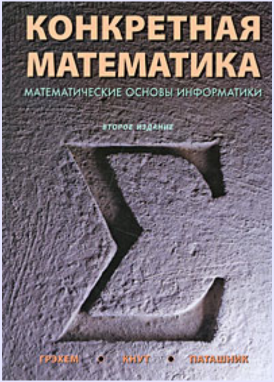
Original slides 2010–2014 Jaan Penjam; modified 2016–2020 Silvio Capobianco

CONtinuous + disCRETE
MATHEMATICS

The book





http://www-cs-faculty.stanford.edu/~uno/gkp.html

Concrete Mathematics is ...

- the controlled manipulation of mathematical formulas
- using a collection of techniques for solving problems

Goals of the book:

- to introduce the mathematics that supports advanced computer programming and the analysis of algorithms
- to provide a solid and relevant base of mathematical skills - the skills needed
  - to solve complex problems
  - to evaluate horrendous sums
  - to discover subtle patterns in data

TAL TECH

Our additional goals

- to get acquainted with well-known and popular literature in CS and Math
- to develop mathematical skills, formulating complex problems mathematically
- to practice presentation of results (solutions of mathematical problems)

Contents of the Book

Chapters:

Recurrent problems

Recurrence equations

A sequence of complex numbers $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ is called recurrent, if for $n \geq 1$ its generic term a_n satisfies a recurrence equation

$$a_n = f_n(a_{n-1}, \ldots, a_0),$$

where $f_n : \mathbb{C}^n \to \mathbb{C}$ for every $n \geq 1$.
If there exists $f : \mathbb{N} \times \mathbb{C}^k \to \mathbb{C}$ such that:

$$f_n = f(n; a_{n-1}, \ldots, a_{n-k}) \text{ for every } n \geq k,$$

the number k is called the order of the recurrence equation.

recurrent (<Latin *recurrere* – to run back) tagasipöörduv, taastuv / to run back.

Two examples of recurrence equations

A recurrence equation of order 2

$$
\begin{aligned}
a_0 &= 0\,; a_1 = 1\,; \\
a_n &= a_{n-1} + a_{n-2} \text{ for every } n \geq 2
\end{aligned}
$$

This recurrence defines the Fibonacci numbers.

A recurrence equation without a well-defined order

$$
\begin{aligned}
a_0 &= 1\,; \\
a_n &= a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-1} a_0 \text{ for every } n \geq 1
\end{aligned}
$$

This recurrence defines the Catalan numbers.

Motivation: why solve recursions?

- Computing by *closed form* is effective:

$$P_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \left[1 + \frac{1}{12n} + \frac{1}{228n^2} + \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right)\right]$$

- *Closed form* allows to analyze a function using "classical" techniques.

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- *Closed form* allows to analyze a function using "classical" techniques.
  For example: the behavior of the logistic map depends on r:

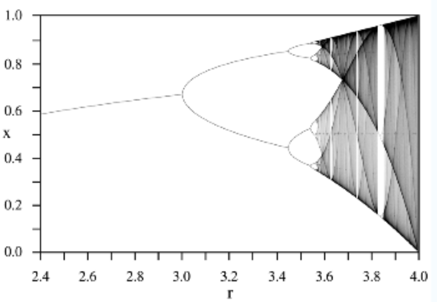
Recurrence equation:

$$x_{n+1} = rx_n(1 - x_n)$$

Solution for $r = 4$:

$$x_{n+1} = \sin^2(2^n \theta \pi)$$

with $\theta = \frac{1}{\pi}\arcsin(\sqrt{x_0})$



ad hoc techniques: Guess and Confirm

Equation $f(n) = (n^2 - 1 + f(n-1))/2$, initial condition: $f(0) = 2$

- Let's compute some values:

n	0	1	2	3	4	5	6
$f(n)$	2	1	2	5	10	17	26

Guess: $f(n) = (n-1)^2 + 1$.

- Assuming that the guess holds for $n = k$, we prove that it holds for $n = k+1$:

$$\begin{aligned} f(k+1) &= ((k+1)^2 - 1 + f(k))/2 \\ &= (k^2 + 2k + (k-1)^2 + 1)/2 \\ &= (k^2 + 2k + k^2 - 2k + 1 + 1)/2 \\ &= (2k^2 + 2)/2 = k^2 + 1 \end{aligned}$$

QED.

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\begin{aligned}
f(k+1) &= ((k+1)^2 - 1 + f(k))/2 \\
&= (k^2 + 2k + (k-1)^2 + 1)/2 \\
&= (k^2 + 2k + k^2 - 2k + 1 + 1)/2 \\
&= (2k^2 + 2)/2 = k^2 + 1
\end{aligned}
$$

QED.

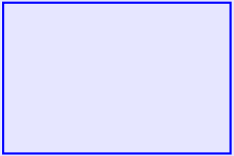
TAL TECH

Next section

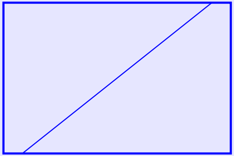
1. Recurrent Problems

1. The Tower of Hanoi
2. Lines in the Plane
3. The Josephus Problem

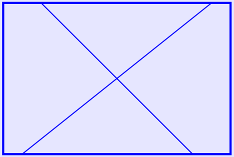
Regions of the plane defined by lines



$Q_0 = 1$



$Q_1 = 2$

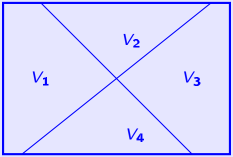


$Q_2 = 4$

In general: $Q_n = 2^n$?

Regions of the plane defined by lines

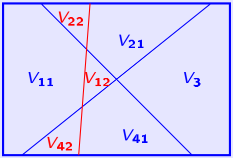
Actually ...



$$Q_2 = 4$$

Regions of the plane defined by lines

Actually ...



$$Q_3 = Q_2 + 3 = 7$$

Generally $Q_n = Q_{n-1} + n$.

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
Q_n	1	2	4	7	11	16	22	29	37	46	$\cdots$

Regions of the plane defined by lines

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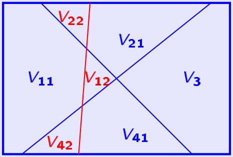

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TAL TECH

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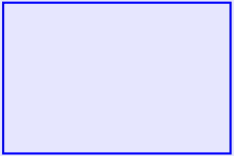
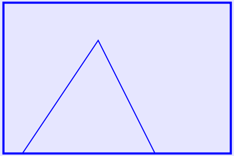
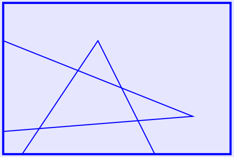


$T_0 = 1$



$T_1 = 2$

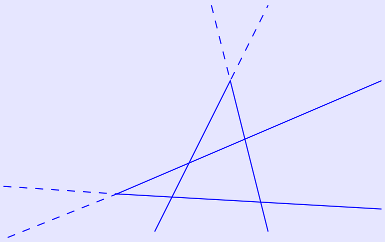


$T_2 = 7$

$T_3 = ?$

$T_n = ?$

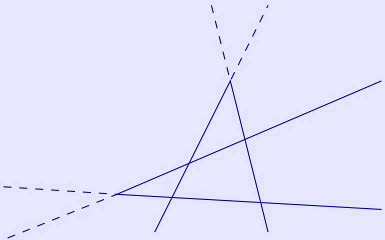
Regions of the plane defined by lines



$$
\begin{aligned}
T_2 &= Q_4 - 2 \cdot 2 = 11 - 4 = 7 \\
T_3 &= Q_6 - 2 \cdot 3 = 22 - 6 = 16 \\
T_4 &= Q_8 - 2 \cdot 4 = 37 - 8 = 29 \\
T_5 &= Q_{10} - 2 \cdot 5 = 56 - 10 = 46 \\
\\
T_n &= Q_{2n} - 2n
\end{aligned}
$$

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
Q_n	1	2	4	7	11	16	22	29	37	46	$\cdots$
T_n	1	2	7	16	29	46	67	92	121	156	$\cdots$

Regions of the plane defined by lines



$$T_2 = Q_4 - 2 \cdot 2 = 11 - 4 = 7$$
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n	0	1	2	3	4	5	6	7	8	9	$\cdots$
Q_n	1	2	4	7	11	16	22	29	37	46	$\cdots$
T_n	1	2	7	16	29	46	67	92	121	156	$\cdots$

Next section

2. Sums

Sums as solutions of recurrences

The simplest recurrences have the form:

$$
\begin{aligned}
a_0 &= c_0 \,; \\
a_n &= a_{n-1} + c_n \text{ for every } n \geq 1 \,.
\end{aligned}
$$

The solution to the above is clearly:

$$a_n = \sum_{k=0}^{n} c_k$$

Problem: find a closed form for the sum!

Finite calculus

A way of “working on sums like they were integrals”:

- Finite difference instead of derivative:

$$\Delta f(x) = f(x-1) - f(x) \text{ for every } x$$

- A new family of *elementary functions* which solve specific difference equations (instead of “differential”):
  - Falling factorials in place of powers.
  - Harmonic numbers in place of logarithm.
- “Summation by parts”.
- Stolz-Cesàro lemma in place of l'Hôpital's rule.

Infinite sums

On the one hand:

Example 1

Let
$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$
Then
$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$
and
$$S = 2$$

Infinite sums

...but on the other hand:

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \dots = T - 1$$

and

$$T = -1$$

Next section

3. Integer Functions

Some important functions with integer values

The Iverson brackets which "translate booleans into integers, with a twist":

1. $[\text{True}] = 1$ and $[\text{False}] = 0$.
2. If a is infinite or undefined, then $a \cdot [\text{False}] = 0$.

The ceiling of a real number:

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid x \leq n\}$$

and its "dual", the floor:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$$

Next section

4. Number Theory

Next section

TAL TECH

5. Binomial Coefficients

Next section

6. Special Numbers

TAL TECH

Next section

7. Generating Functions

Solving recurrences with generating functions

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n which expresses it as a function of n, but not of $g_0, \ldots, g_{n-1}$.

The method of generating functions

1. Write down a single equation that expresses g_n in terms of other elements of the sequence. This equation should be valid for all integers n, assuming that $g_{-1} = g_{-2} = \ldots = 0$.
2. Multiply both sides of the equation by z^n and sum over all n.
   This gives, on the left-hand side, the series $\sum_n g_n z^n$, which is the generating function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
3. Solve the resulting equation, getting a closed form for $G(z)$.
4. Expand $G(z)$ into a power series and read off the coefficient of z^n: thanks to the properties of analytic functions in the complex plane, this is a closed form for g_n.

Example: Fibonacci numbers

1. Single equation holding for every $n \in \mathbb{Z}$:

$$g_n = g_{n-1} + g_{n-2} + (\text{if } n = 1 \text{ then } 1 \text{ else } 0)$$

2. Multiply by z^n and obtain an equation for $G(z) = \sum_n g_n z^n$:

$$G(z) = zG(z) + z^2 G(z) + z$$

3. Solve with respect to $G(z)$:

$$G(z) = \frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi z} - \frac{1}{1 - \hat{\phi} z} \right) \text{ where } \phi = \frac{1 + \sqrt{5}}{2}, \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

4. Derive an expression for g_n which only depends on n:

$$g_n = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) \text{ for every } n \geq 0$$

TAL TECH

Next section

8. Discrete Probability

1. Definitions
2. Mean and Variance
3. Probability Generating Functions
4. Flipping Coins
5. Hashing

Next section

9. Asymptotics

Pedagogical dilemma: what to teach?

Chapters:

1. Recurrent Problems
2. Sums
3. Integer Functions
4. Number Theory
5. Binomial Coefficients
6. Special Numbers
7. Generating Functions
8. Discrete Probability
9. Asymptotics

Last edition's program

- Week 1: Introduction
- Weeks 2 and 3: Recurrent Problems
- Weeks 3, 4 and 5: Sums
- Week 5: Integer Functions
- (Week 6: Winter School in Computer Science)
- Weeks 7 and 8: Number Theory
- Weeks 9 and 10: Binomial Coefficients
- Weeks 11 and 12: Special Numbers
- Weeks 13 and 14: Generating Functions
- Weeks 15 and 16: Asymptotics

Grading

Based on 100 points, distributed as follows:

- Two classroom presentations: 10 points each.
- A midterm test: 30 points.
- The final exam: 50 points.

The final grade G is computed from the total score S as follows:

- 91 or more: 5.
- 81 to 90: 4.
- 71 to 80: 3.
- 61 to 70: 2.
- 51 to 60: 1.
- 50 or less: 0.

Grading

Based on 100 points, distributed as follows:

- Two classroom presentations: 10 points each.
- A midterm test: 30 points.
- The final exam: 50 points.

The prerequisites to be admitted to the final exam are:

1. Attendance to more than half of lectures and exercises.
   (8 lectures and 8 exercise sessions in the next 15 weeks)
2. At least one classroom presentation.
3. At least 15 points at the midterm test.

Contact

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