# ITB8832 Mathematics for Computer Science
# Exercise session 1: 6 September 2023

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## Problems from Section 1.1

### Problem 1.5.

Albert announces to his class that he plans to surprise them with a quiz sometime next week.

His students first wonder if the quiz could be on Friday of next week. They reason that it can't: if Albert didn't give the quiz before Friday, then by midnight Thursday, they would know the quiz had to be on Friday, and so the quiz wouldn't be a surprise any more.

Next the students wonder whether Albert could give the surprise quiz Thursday. They observe that if the quiz wasn't given before Thursday, it would have to be given on the Thursday, since they already know it can't be given on Friday. But having figured that out, it wouldn't be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can't be on Wednesday, Tuesday, or Monday. Namely, it's impossible for Albert to give a surprise quiz next week. All the students now relax, having concluded that Albert must have been bluffing. And since no one expects the quiz, that's why, when Albert gives it on Tuesday next week, it really is a surprise!

What, if anything, do you think is wrong with the students' reasoning?

### Problem 1.2.

What's going on here?!

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (\sqrt{(-1)})^2 = -1\,.$$

(a) Precisely identify and explain the mistake(s) in this *bogus* proof.

(b) Prove (correctly) that if $1 = -1$, then $2 = 1$.

## Problem 1.3.

Identify the bugs in the following bogus proofs:

(a) **Bogus claim:** $1/8 > 1/4$.
*Bogus proof:*

$$\begin{aligned} 3 &> 2 \\ 3\log_{10}(1/2) &> 2\log_{10}(1/2) \\ \log_{10}(1/2)^3 &> \log_{10}(1/2)^2 \\ (1/2)^3 &> (1/2)^2 \end{aligned}$$

and the claim now follows by the rules for multiplying fractions.

(b) *Bogus proof:* $1c = \$0.01 = (\$0.1)^2 = (10c)^2 = 100c = \$1$.

(c) **Bogus claim:** If $a$ and $b$ are two equal real numbers, then $a = 0$.

*Bogus proof:*

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= (a-b)b \\ a+b &= b \\ a &= 0 \end{aligned}$$

## Problem 1.4

The *arithmetic-geometric inequality* states that the arithmetic mean of two nonnegative numbers is an upper bound to their geometric mean, that is:

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a, b \geq 0\,.$$

However, there is something questionable about the following proof of this fact. What is the objection, and how would you fix it?

*Bogus proof:*

$$\begin{aligned}
\frac{a+b}{2} &\stackrel{?}{\geq} \sqrt{ab} && \text{so} \\
a+b &\stackrel{?}{\geq} 2\sqrt{ab} && \text{so} \\
a^2+2ab+b^2 &\stackrel{?}{\geq} 4ab && \text{so} \\
a^2-2ab+b^2 &\stackrel{?}{\geq} 0 && \text{so} \\
(a-b)^2 &\stackrel{?}{\geq} 0 && \text{which we know is true.}
\end{aligned}$$

The last statement is true because $a-b$ is a real number, and the square of a real number is never negative. This proves the claim.

# Problems for Section 1.5

## Problem 1.6.

Show that $\log_7 n$ is either an integer or irrational, where $n$ is a positive integer. Use whatever familiar facts about integers and primes you need, but explicitly state such facts.

# Problems for Section 1.7

## Problem 1.7.

Prove by cases that

$$\max(r,s)+\min(r,s)=r+s$$

for all real numbers $r, s$.

## Problem 1.8.

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

## Problem 1.9.

Prove by cases that $|r+s| \leq |r|+|s|$ for any two real numbers $r$, $s$.

## Problem 1.10(a).

Suppose that

$$a + b + c = d\,,$$

where $a$, $b$, $c$, $d$ are nonnegative integers.

Let $P$ the assertion that $d$ is even. Let $W$ be the assertion that exactly one among $a$, $b$, and $c$ is even, and let $T$ be the assertion that all three are even.

Prove by cases that:

$$P \textbf{ iff } (W \textbf{ or } T)\,.$$

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# Solutions

## Problem 1.5.

The students are trying to prove that the statement "Albert will surprise us with a quiz next week" is false.

First and foremost, this is not what Albert said! Albert said that he *planned* to surprise them: not that he would do so. The students are trying to solve a problem which is a wrong version of the actual one.

But most important, the statement "Albert will surprise us with a quiz next week" *is not a proposition:* it relates to a future event, which may or may not happen, so it does not have a definite truth value *now.*

Albert's statement, on the other hand, *is* a proposition: it says that Albert plans to do something, and that is true!

## Problem 1.2.

(a) The first two equalities are correct.

The third equality is *seriously* wrong! First of all, it *changes the context:* negative real numbers do not have a real square root, so we are moving from real numbers to complex numbers. But this creates a new problem, because *in the complex field, the law of square roots does not hold:* in general, $\sqrt{ab}$ is different from $\sqrt{a}\sqrt{b}$, because every nonzero complex number has two square roots, not one. For example, if we choose the value $i$ for the first square root of $-1$ and $-i$ for the second one, then their product is 1, not $-1$.

The fourth equality is also wrong, because we are not sure if the square root of $-1$ which we chose for the first factor in the left-hand side is the same as the one we chose for the second factor.

The last equality is correct: $\sqrt{-1}$, however we choose it, is one of the two solutions of the equation $x^2 = -1$.

(b) There are several options; the one from today's (6 September 2023) session goes as follows. Suppose $1 = -1$. By adding 1 to both sides, $2 = 0$. By multiplying both sides by $3/2$, $3 = 0$. By subtracting 1 from both sides, $2 = -1$. By applying the equality $1 = -1$, $2 = 1$.

## Problem 1.3.

(a) The error is in the second inequality. If the base is larger than 1 and the argument is smaller than 1, then the logarithm is negative: hence, the multiplication factor $\log_{10}(1/2)$ is negative. But multiplying by a negative quantity reverses the sign of the inequality, which has not been done.

(b) The error is in the wrong use of measure units. One dollar is the square of *one square root of a dollar*; similarly, one cent is the square of one square root of a cent. In turn, if one dollar corresponds to a hundred cents, then one square root of dollar corresponds to ten square root of cents, and 0.1 square root of dollars correspond to one square root of cent. The correct chain of equalities is thus:

$$1c = \$0.01 = (\sqrt{\$}\,0.1)^2 = (1\sqrt{c})^2 = 1c\,.$$

(c) Everything is fine until the second last equality. If $a = b$, then $a - b = 0$, and we cannot simplify the previous equality to get it.

## Problem 1.4

The argument above is not a proof of the arithmetic-geometric inequality! Written as we did, it is *a proof of* $(a-b)^2 \geq 0$ *given the arithmetic-geometric inequality*. That is: we are going in the *wrong direction*.

To solve the issue, we observe that each inequality is *equivalent* to the next one: however given $a$ and $b$, either they are both verified, or neither is. A key point here is that $a$ and $b$ are nonnegative reals: in particular, the critical passage from $a^2 + 2ab + b^2 \geq 4ab$ to $a + b \geq \sqrt{ab}$ is valid. We can thus replace "so" with "which is equivalent to" in the argument above, and get a proof of the arithmetic-geometric inequality.

## Problem 1.6.

For what we know now, $\log_7 n$ could be either an integer, or a noninteger rational, or an irrational. The thesis is then equivalent to saying that the second case never happens: that is, if $\log_7 n$ is rational, then it is integer. If $n = 1$ then $\log_7 n = 0$ is integer, so we can assume $n \geq 2$.

Suppose that we can write $\log_7 n = a/b$ with $a$ and $b$ integers; as $n \geq 2$ and $7 > 1$, we may suppose $a$ and $b$ both positive. By definition of logarithm, $n = 7^{a/b}$; by taking the $b$th powers, $n^b = 7^a$.

Now we recall that if a prime number id a divisor of a product of integers, then it is a divisor of at least one of the factors. As the right-hand side is a power of 7 larger than 1 (because $a > 0$) and the left-hand side is a product of $b$ factors all equal to $n$, $n$ must be a multiple of 7. Let then $k$ be the largest positive integer such that $n$ is divisible by $7^k$: then $n = 7^k m$ for some positive integer $m$ which is *not* a multiple of 7. But the equality $7^a = n^b = 7^{kb} m^b$ can hold only if $m = 1$ (because any prime factor of $m$ should be a divisor of 7, which is impossible) and $kb = a$: then $k = a/b = \log_7 n$ is integer, as we wanted to prove.

## Problem 1.7.

Exactly three cases are possible: $r > s$, $r = s$, or $r < s$. Let us consider them one by one:

$r > s$. In this case, $\max(r, s) = r$ and $\min(r, s) = s$, so the equality becomes $r + s = r + s$, which is trivially true.

$r = s$. In this case, the maximum and minimum of $r$ and $s$ are both equal to the common value $t$ of $r$ and $s$, so the equality becomes $2t = 2t$, which is trivially true.

$r < s$. In this case, $\max(r, s) = s$ and $\min(r, s) = r$, so the equality becomes $s + r = r + s$, which is true by the commutative property.

## Problem 1.8.

We must find two irrational numbers $a$ and $b$ such that $a^b$ is rational. We are given $\sqrt{2}^{\sqrt{2}}$, of which we don't know if it is rational or irrational[1]. However, we know that it is *one of the two*, so we can proceed by cases:

1. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we can just choose $a = b = \sqrt{2}$.
2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, we make an experiment and raise it to the power $b = \sqrt{2}$. Then:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2\,,$$

   which is rational. We can then take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

At the end of this discussion, let's remark that the proof we gave is *nonconstructive*, in the sense that we have not given a *witness* of the proposition "there exist two irrational numbers $a$ and $b$ such that $a^b$ is rational"; in our case, such witness would be the pair $(a, b)$. What we gave instead, is:

[1] It is actually irrational by the *Gelfond-Schneider theorem*.

1. a witness of the proposition "if $\sqrt{2}^{\sqrt{2}}$ is rational, then there exist two irrational numbers $a$ and $b$ such that $a^b$ is rational", such witness being the pair $(\sqrt{2}, \sqrt{2})$; and

2. a witness of the proposition "if $\sqrt{2}^{\sqrt{2}}$ is irrational, then there exist two irrational numbers $a$ and $b$ such that $a^b$ is rational", such witness being the pair $\left(\sqrt{2}^{\sqrt{2}}, \sqrt{2}\right)$.

But to conclude that the original proposition "there exist two irrational numbers $a$ and $b$ such that $a^b$ is rational" is true, we needed to use the fact that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational: which is an application of the law of excluded middle.

## Problem 1.9.

**Solution.** Recall the definition: if r is a real number, then $|r|$ equals $r$ if $r \geq 0$, and $-r$ if $r < 0$. Equivalently: $|r| = \max(r, -r)$.

For two real numbers $r$ and $s$, four cases are possible:

1. $r \geq 0$ and $s \geq 0$. In this case, $r + s \geq 0$ too, so:
$$|r + s| = r + s = |r| + |s| \ .$$

2. $r < 0$ and $s < 0$. In this case, $r + s < 0$ too, so:
$$|r + s| = -(r + s) = -r - s = |r| + |s| \ .$$

3. $r \geq 0$ and $s < 0$. In this case, on the one hand:
$$r + s < r = |r| \leq |r| + |s| \ ;$$
and on the other hand:
$$-(r + s) = -r + |s| \leq |r| + |s| \ .$$

4. $r < 0$ and $s \geq 0$. In this case, we can either deason as in the previous point, or observe that, since addition is commutative and the inequality is true for *every* $r \geq 0$ and $s < 0$, it is also true with *the role of* $r$ *and* $s$ *swapped.*

## Problem 1.10(a).

There are at least two ways of choosing a "good" set of cases. In both cases (pardon) the rule "even plus even is even, even plus odd is odd, odd plus odd is even" will play a key role. As addition of nonnegative integers is commutative, we can reason only on the *number* of summands which are odd.

1. *First option: cases determined by truth values of $W$ and $T$.*

   (a) If $W$ is true, then $a+b+c$ is the sum of one even and two odd summands, so $d$ is even, and $P$ is true. Also, if $W$ is true, then so is $W$ **or** $T$.

   (b) If $T$ is true, then $a+b+c$ is the sum of three even summands, so $d$ is even. Also, if $T$ is true, then so is $W$ **or** $T$.

   (c) If $W$ and $T$ are both false, then the number of even summands in the sum $a+b+c$ is neither one nor three, so it is either zero or two: these correspond to one or three odd summands, respectively. A sum where oddly more summands are odd is odd, so in this case, $d$ is odd, and $P$ is false. Also, $W$ **or** $T$ is false.

   Summarizing: if either $W$ or $T$ is true, then $P$ is true, and if $W$ and $T$ are both false, then $P$ is false. This concludes the proof.

2. *Second option: cases determined by number of even summands on the left-hand side.*

   (a) If all three summands are even, then $T$ is true and so is $P$.

   (b) If one summand is odd, then $W$ and $T$ are both false, and so is $P$.

   (c) If two summands are odd, then $W$ is true, and so is $P$.

   (d) If all three summands are odd, then $W$ and $T$ are both false, and so is $P$.

   Summarizing: if either $W$ or $T$ is true, then $P$ is true, and if $P$ is true, then either $W$ or $P$ is true. This concludes the proof.