ITB8832 Mathematics for Computer Science
Exercise session 1: 4 September 2024

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Problems from Section 1.1

Problem 1.5.

Albert announces to his class that he plans to surprise them with a quiz sometime next week.

His students first wonder if the quiz could be on Friday of next week. They reason that it can't: if Albert didn't give the quiz before Friday, then by midnight Thursday, they would know the quiz had to be on Friday, and so the quiz wouldn't be a surprise any more.

Next the students wonder whether Albert could give the surprise quiz Thursday. They observe that if the quiz wasn't given before Thursday, it would have to be given on the Thursday, since they already know it can't be given on Friday. But having figured that out, it wouldn't be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can't be on Wednesday, Tuesday, or Monday. Namely, it's impossible for Albert to give a surprise quiz next week. All the students now relax, having concluded that Albert must have been bluffing. And since no one expects the quiz, that's why, when Albert gives it on Tuesday next week, it really is a surprise!

What, if anything, do you think is wrong with the students' reasoning?

Problem 1.2.

What's going on here?!

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (\sqrt{(-1)})^2 = -1\,.$$

(a) Precisely identify and explain the mistake(s) in this *bogus* proof.

(b) Prove (correctly) that if $1 = -1$, then $2 = 1$.

Problem 1.3.

Identify the bugs in the following bogus proofs:

(a) **Bogus claim:** $1/8 > 1/4$.

Bogus proof:

$$\begin{array}{rcl} 3 & > & 2 \\ 3\log_{10}(1/2) & > & 2\log_{10}(1/2) \\ \log_{10}(1/2)^3 & > & \log_{10}(1/2)^2 \\ (1/2)^3 & > & (1/2)^2 \end{array}$$

and the claim now follows by the rules for multiplying fractions.

(b) *Bogus proof:* $1c = \$0.01 = (\$0.1)^2 = (10c)^2 = 100c = \$1$.

(c) **Bogus claim:** If a and b are two equal real numbers, then $a = 0$.

Bogus proof:

$$\begin{array}{rcl} a & = & b \\ a^2 & = & ab \\ a^2 - b^2 & = & ab - b^2 \\ (a-b)(a+b) & = & (a-b)b \\ a+b & = & b \\ a & = & 0 \end{array}$$

Problem 1.4

The *arithmetic-geometric inequality* states that the arithmetic mean of two nonnegative numbers is an upper bound to their geometric mean, that is:

$$\frac{a+b}{2} \geq \sqrt{ab} \;\; \forall a, b \geq 0\,.$$

However, there is something questionable about the following proof of this fact. What is the objection, and how would you fix it?

Bogus proof:

$$\begin{aligned}
\frac{a+b}{2} &\overset{?}{\geq} \sqrt{ab} && \text{so} \\
a+b &\overset{?}{\geq} 2\sqrt{ab} && \text{so} \\
a^2+2ab+b^2 &\overset{?}{\geq} 4ab && \text{so} \\
a^2-2ab+b^2 &\overset{?}{\geq} 0 && \text{so} \\
(a-b)^2 &\overset{?}{\geq} 0 && \text{which we know is true}.
\end{aligned}$$

The last statement is true because $a - b$ is a real number, and the square of a real number is never negative. This proves the claim.

Problems for Section 1.5

Problem 1.6.

Show that $\log_7 n$ is either an integer or irrational, where n is a positive integer. Use whatever familiar facts about integers and primes you need, but explicitly state such facts.

Problems for Section 1.7

Problem 1.7.

Prove by cases that

$$\max(r, s) + \min(r, s) = r + s$$

for all real numbers r, s.

Problem 1.8.

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

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Solutions

Problem 1.5.

The students are trying to prove that the statement "Albert will surprise us with a quiz next week" is false.

First and foremost, this is not what Albert said! Albert said that he *planned* to surprise them: not that he would do so. The students are trying to solve a problem which is a wrong version of the actual one.

More importantly, the statement "Albert will surprise us with a quiz next week" *is not a proposition:* it relates to a future event, which may or may not happen, so it does not have a definite truth value *now*.

There is at least one more observation that can be made.[1] What the students actually (think they have) proved, is that there will not be a *surprise* test next week: that is, *if there will be a test, then it will not be a surprise.* But this doesn't mean that there won't be any test at all!

Albert's statement, on the other hand, *is* a proposition: it says that Albert plans to do something, and that is true!

Problem 1.2.

(a) The first two equalities are correct.

The third equality is *seriously* wrong! First of all, it *changes the context:* negative real numbers do not have a real square root, so we are moving from real numbers to complex numbers. But this creates a new problem, because *in the complex field, the law of square roots does not hold:* in general, $\sqrt{ab}$ is different from $\sqrt{a}\sqrt{b}$, because every nonzero complex number has two square roots, not one. For example, if we choose the value i for the first square root of -1 and $-i$ for the second one, then their product is 1, not -1.

The fourth equality is also wrong, because we are not sure if the square root of -1 which we chose for the first factor in the left-hand side is the same as the one we chose for the second factor.

The last equality is correct: $\sqrt{-1}$, however we choose it, is one of the two solutions of the equation $x^2 = -1$.

(b) There are several options:

[1] Suggested in classroom on 4 September 2024.

- Suppose $1 = -1$. By adding 1 to both sides, $2 = 0$. By multiplying both sides by $3/2$, $3 = 0$. By subtracting 1 from both sides, $2 = -1$. By applying the equality $1 = -1$, $2 = 1$.
- Suppose $1 = -1$. By adding 1 to both sides, $2 = 0$. By squaring both sides, $4 = 0$. By the two previous equalities, $4 = 2$. By dividing by 2, $2 = 1$.

Problem 1.3.

(a) The error is in the second inequality. If the base is larger than 1 and the argument is smaller than 1, then the logarithm is negative: hence, the multiplication factor $\log_{10}(1/2)$ is negative. But multiplying by a negative quantity reverses the sign of the inequality, which has not been done.

(b) The error is in the wrong use of measure units. One dollar is the square of *one square root of a dollar*; similarly, one cent is the square of one square root of a cent. In turn, if one dollar corresponds to a hundred cents, then one square root of dollar corresponds to ten square root of cents, and 0.1 square root of dollars correspond to one square root of cent. The correct chain of equalities is thus:

$$1c = \$0.01 = (\sqrt{\$}\,0.1)^2 = (1\sqrt{c})^2 = 1c\,.$$

(c) Everything is fine until the second last equality. If $a = b$, then $a - b = 0$, and we cannot simplify the previous equality to get it.

Problem 1.4

The argument above is not a proof of the arithmetic-geometric inequality! Written as we did, it is *a proof of* $(a-b)^2 \geq 0$ *given the arithmetic-geometric inequality*. That is: we are going in the *wrong direction*.

To solve the issue, we observe that each inequality is *equivalent* to the next one: however given a and b, either they are both verified, or neither is. A key point here is that a and b are nonnegative reals, because any inequality between nonnegative reals is equivalent to the same inequality between their squares. In particular, the critical passage from $a^2 + 2ab + b^2 \geq 4ab$ to $a + b \geq \sqrt{ab}$ is valid. We can thus replace "so" with in the argument above, and get a proof of the arithmetic-geometric inequality.

Problem 1.6.

For what we know now, $\log_7 n$ could be either an integer, or a noninteger rational, or an irrational. The thesis is then equivalent to saying that the second case never happens: that is, if $\log_7 n$ is rational, then it is integer.

Suppose that $\log_7 n = a/b$ with a and b integers. As $n \geq 2$ and $7 > 1$, we may suppose a and b both positive. By definition of logarithm, $n = 7^{a/b}$; by taking the bth powers, $n^b = 7^a$.

Now we recall that the following fundamental property of prime numbers: if a prime number is a divisor of a product of integers, then it is a divisor of at least one of the factors. As the left-hand side is a product of a factors, all equal to n, and the right-hand side is a power of 7, the only prime that divides n must be 7, that is, n must be a power of 7. Let then k be the unique integer such that $n = 7^k$ is divisible by 7^k: then $n = 7^k m$ for some positive integer m which is *not* a multiple of 7. But the equality $n^b = 7^{kb} = 7^a$ is equivalent to $a = kb$, so $k = a/b = \log_7 n$ is integer, as we wanted to prove.

Problem 1.7.

Exactly three cases are possible: $r > s$, $r = s$, or $r < s$. Let us consider them one by one:

$r > s$. In this case, $\max(r, s) = r$ and $\min(r, s) = s$, so the equality becomes $r + s = r + s$, which is trivially true.

$r < s$. In this case, $\max(r, s) = s$ and $\min(r, s) = r$, so the equality becomes $s + r = r + s$, which is true by the commutative property.

$r = s$. In this case, the maximum and minimum of r and s are both equal to the common value t of r and s, so the equality becomes $2t = 2t$, which is trivially true.

Problem 1.8.

We must find two irrational numbers a and b such that a^b is rational. We are given $\sqrt{2}^{\sqrt{2}}$, of which we don't know if it is rational or irrational.[2] However, we know that it is *one of the two*, so we can proceed by cases:

1. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we can just choose $a = b = \sqrt{2}$.

[2] It is actually irrational by the *Gelfond-Schneider theorem*.

2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, we make an experiment and raise it to the power $b = \sqrt{2}$. Then:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{2}\right)^{2} = 2\,,$$

which is rational. We can then take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

At the end of this discussion, let's remark that the proof we gave is *nonconstructive*, in the sense that we have not given a *witness* of the proposition "there exist two irrational numbers a and b such that a^b is rational"; in our case, such witness would be the pair (a, b). What we gave instead, is:

1. a witness of the proposition "if $\sqrt{2}^{\sqrt{2}}$ is rational, then there exist two irrational numbers a and b such that a^b is rational", such witness being the pair $\left(\sqrt{2}, \sqrt{2}\right)$; and

2. a witness of the proposition "if $\sqrt{2}^{\sqrt{2}}$ is irrational, then there exist two irrational numbers a and b such that a^b is rational", such witness being the pair $\left(\sqrt{2}^{\sqrt{2}}, \sqrt{2}\right)$.

But to conclude that the original proposition "there exist two irrational numbers a and b such that a^b is rational" is true, we needed to use the fact that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational: which is an application of the law of excluded middle.