Mathematics for Computer Science
Exercise session 2, 13 September 2023

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Problems for Section 1.8

Problem 1.17.

Prove that $\log_4 6$ is irrational.

Problem 1.18.

Prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational.
Hint: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$.

Problem 1.22.

A familiar proof that $\sqrt[3]{7^2}$ is irrational depends on the fact that a certain equation among those below is unsatisfiable by integers $a, b > 0$. (Note that more than one is unsatisfiable, but only one of them is relevant.) solutions with both a and b positive integers. Indicate the equation that would appear in the proof, and explain why it is unsatisfiable. (Do *not* assume that $\sqrt[3]{7^2}$ is irrational.)

1. $a^2 = 7^2 + b^2$.
2. $a^3 = 7^2 + b^3$.
3. $a^2 = 7^2 b^2$.
4. $a^3 = 7^2 b^3$.
5. $a^3 = 7^3 b^3$.
6. $(ab)^3 = 7^2$.

Problems from Section 2.2

Problem 2.2 (with some small changes).

The *Fibonacci numbers* $F(0), F(1), F(2), \ldots$ are defined as follows:

$$F(n) = \begin{cases} 0 & \text{if } n = 0\,, \\ 1 & \text{if } n = 1\,, \\ F(n-1) + F(n-2) & \text{if } n > 1\,, \end{cases}$$

Exactly which sentence(s) in the following bogus proof contain logical errors? Explain.

Theorem (Bogus theorem)**.** *Every Fibonacci number is even.*

Bogus proof. Let all the variables n, m, k mentioned below be nonnegative integer valued.

1. Let $EF(n)$ mean that $F(n)$ is even.
2. Let C be the set of counterexamples to the assertion that $EF(n)$ holds for all $n \in \mathbb{N}$, namely,
$$C ::= \{n \in \mathbb{N} \mid \textbf{not}(EF(n))\}\,.$$
3. Assume C is nonempty. By WOP, it has a minimum m.
4. Then $m > 0$, since $F(0) = 0$ is an even number.
5. Since m is a minimum counterexample, $F(k)$ is even for all $k < m$.
6. In particular, $F(m-1)$ and $F(m-2)$ are both even.
7. But $F(m) = F(m-1) + F(m-2)$, and the right-hand side is even.
8. That is, $EF(m)$ is true, and m is not a true counterexample.
9. Then C is empty, and $F(n)$ is even for all $n \in \mathbb{N}$.

Problem 2.4.

Use the *Well Ordering Principle* to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

for all nonnegative integers n.

Problem 2.5

Use the Well Ordering Principle to prove that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3 \,.$$

Problems for Section 2.4

Problem 2.23.

Prove that a set R of real numbers is well ordered iff there is no infinite decreasing sequence of numbers in R. In other words: R there is no set of numbers $r_i \in R$ such that

$$r_0 > r_1 > r_2 > \dots \tag{2}$$

Problems for Section 3.1

Problem 3.2.

Your class has a textbook and a final exam. Let P, Q, and R be the following propositions:

- P ::= "You get an A on the final exam."
- Q ::= "You do every exercise in the book."
- R ::= "You get an A in the class."

Translate following assertions into propositional formulas using P, Q, R, and the propositional connectives **and** , **not**(), **implies** .

(a) You get an A in the class, but you do not do every exercise in the book.

(b) You get an A on the final exam, you do every exercise in the book, and you get an A in the class.

(c) To get an A in the class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

Problem 3.5.

Sloppy Sam is trying to prove a certain proposition P. He defines two related propositions Q and R, and then proceeds to prove three implications:

$$P \textbf{ implies } Q\,, \quad Q \textbf{ implies } R\,, \quad R \textbf{ implies } P\,.$$

He then reasons as follows:

> If Q is true, then since I proved Q **implies** R, I can conclude that R is true. Now, since I proved R **implies** P, I can conclude that P is true. Similarly, if R is true, then P is true and so Q is true. Likewise, if P is true, then so are Q and R. So any way you look at it, all three of P, Q and R are true.

(a) Exhibit truth tables for

$$(P \textbf{ implies } Q) \textbf{ and } (Q \textbf{ implies } R) \textbf{ and } (R \textbf{ implies } P) \tag{3}$$

and for

$$P \textbf{ and } Q \textbf{ and } R\,. \tag{4}$$

Use these tables to find a truth assignment for P, Q, R so that (3) is **T** and (4) is **F**.

(b) You show these truth tables to Sloppy Sam and he says "OK, I'm wrong that P, Q and R all have to be true, but I still don't see the mistake in my reasoning. Can you help me understand my mistake?" How would you explain to Sammy where the flaw lies in his reasoning?

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Solutions

Problem 1.17.

By contradiction, assume $\log_4 6 = m/n$ for suitable positive integers m, n. As we saw during the lecture, we may suppose also that m and n are positive and relatively prime. Then $6^n = 4^{n \log_4 6} = 4^m$, which is impossible, because 6^n is divisible by 3, and 4^m is not.

Important note: As 4 is not prime, we *cannot* conclude that, since 6 is not an integer power of 4, then $\log_4 6$ is irrational. For example, 8 is not an integer power of 4, but $\log_4 8 = \dfrac{\log_2 8}{\log_2 4} = \dfrac{3}{2}$ is rational.

Problem 1.19.

We follow the hint and perform the multiplication:

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1 .$$

This means that $\sqrt{3} - \sqrt{2}$ is the multiplicative inverse of $\sqrt{3} + \sqrt{2}$. By contradiction, assume $\sqrt{3} + \sqrt{2} = m/n$ is rational. Then $\sqrt{3} - \sqrt{2} = n/m$ is rational too, and so is their difference $2\sqrt{2}$. But then, so is $\sqrt{2}$: contradiction.

If, instead of the difference $2\sqrt{2}$, we consider the sum $2\sqrt{3}$, we reach a similar contradiction. Indeed, an argument similar to our proof of the irrationality of $\sqrt{2}$ leads us to the conclusion that $\sqrt{3}$ is irrational.

Problem 1.22.

1. $a^2 = 7^2 + b^2$ is *satisfiable* in the positive integers, so it won't help us prove that $\sqrt[3]{7^2}$ is irrational *because* a certain equation is *un*satisfiable in the positive integers! Specifically, $(7, 24, 25)$ is a *Pythagorean triple*, that is, a triple of positive integers (k, m, n) such that $k^2 + m^2 = n^2$. We could then put $a = 25$ and $b = 24$.

   And even if it had been unsatisfiable, it involves a sum instead of a ratio or a product, so it cannot help in our case.

2. $a^3 = 7^2 + b^3$ is unsatisfiable. However, it gives us no information about the existance of two positive integers a, b such that $7^2 = a^3/b^3$, so this is not the equation we are looking for.

   To see why the equality is unsatisfiable, observe that it is equivalent to $a^3 - b^3 = 7^2$. But $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = 7 \cdot 7$ with a and b

both positive integers is only possible if $a-b=1$ and $a^2+ab+b^2=7^2$, because if a and b are both positive, then:

$$a^2+ab+b^2 > a^2+b^2 \geq a+b > a-b\,.$$

Then $a=b+1$ and:

$$a^2+ab+b^2 = b^2+2b+1+b^2+b+b^2 = 3b^2+3b+1\,;$$

but $3b^2+3b+1=7^2$ is equivalent to $b(b+1)=16$, which has no integer solutions.

3. $a^2=7^2b^2$ is satisfiable by choosing $b>0$ arbitrarily and $a=7b$.

4. $a^3=7^2b^3$ is the equation we are looking for: it means that there are no two positive integers such that $\sqrt[3]{7^2}=a/b$. The reason why this equation has no positive integer solution is that the exponent of 7 in the prime factorization of the left-hand side is a multiple of 3, but the one of the right-hand side isn't.

5. $a^3=7^3b^3$ is satisfiable by choosing $b>0$ arbitrarily and $a=7b$.

6. $(ab)^3=7^2$ is unsatisfiable for the same reason why $a^3=7^2b^3$ is unsatisfiable. However, a and b appear in a product instead of a ratio, so this is not the equation we are looking for.

Problem 2.2 (with some small changes).

The problem is with point 6. Until now, we only know that m is positive: it could well be 1. (It is so indeed, but that's not the point.) But if $m=1$, then $m-2=-1$ is not a natural number; and we have only defined the Fibonacci numbers as a function on the naturals, not on all integers! For what we know, $F(-1)$ might not exist.[1]

Problem 2.4.

First, a note on notation. Let a be an integer, and for each integer $k \geq a$ let x_k be a complex number. Then for $n \geq a$ integer the *sum, for k from a to n, of x_k* is defined as follows:

$$\sum_{k=a}^{a} x_k = x_a\,; \quad \sum_{k=a}^{n} x_k = \left(\sum_{k=a}^{n-1} x_k\right) + x_n \text{ for every } n > a\,.$$

[1]As a curiosity: it is possible to define the Fibonacci numbers on negative integers, and it turns out that it must be $F(-1)=1$. More in general, if n is a positive integer, then $F(-n)=(-1)^{n-1}F(n)$.

This is an example of a *recursive definition*, where the current value is constructed from the previous ones. We will see more of these in later chapters.

Let C be the set of counterexamples to (1), namely,

$$C ::= \left\{ n \in \mathbb{N} \mid \sum_{k=0}^{n} k^2 \neq \frac{n(n+1)(2n+1)}{6} \right\} .$$

If C is nonempty, then it has a minimum element m: such m must be positive, because for $n = 0$ both sides of (1) are zero. Since m is the minumum of C, $m - 1$, which is still a natural number as m is positive, *does* satisfy (1): we then have

$$\sum_{k=0}^{m-1} k^2 = \frac{(m-1)m(2(m-1)+1)}{6} .$$

But then,

$$\begin{aligned}
\sum_{k=0}^{m} k^2 &= \sum_{k=0}^{m-1} k^2 + m^2 \\
&= \frac{(m-1)m(2(m-1)+1)}{6} + m^2 \\
&= \frac{(m^2-m)(2m-1)+6m^2}{6} \\
&= \frac{2m^3-3m^2+m+6m^2}{6} \\
&= \frac{2m^3+3m^2+m}{6} \\
&= \frac{m(2m^2+3m+1)}{6} \\
&= \frac{m(m+1)(2m+1)}{6} :
\end{aligned}$$

that is, m *does* satisfy (1) after all. The contradiction stems from our hypothesis that C be nonempty: hence, C is empty, and (1) holds for every nonnegative integer m.

Problem 2.5

Let c_0 be the smallest positive integer such that positive integers a_0 and b_0 exist such that $4a_0^3 + 2b_0^3 = c_0^3$. We observe that $c_0 > 1$, because the left-hand side must be even: indeed, c_0 itself must be even, so it must be $c_0 = 2c_1$ for some positive integer c_1. We then have:

$$4a_0^3 + 2b_0^3 = 8c_1^3 ,$$

which, dividing by 2, yields:

$$2a_0^3 + b_0^3 = 4c_1^3 \,.$$

Now, the right-hand side is even, so both summands on the left-hand side must be even: this means that b_0 must be even, so we write $b_0 = 2b_1$ for a suitable positive integer h. Again, we get, first, $2a_0^3 + 8b_1^3 = 4c_1^3$, then, dividing by 2,

$$a_0^3 + 4b_1^3 = 2c_1^3 \,.$$

This time, with the same logic, $a_0 = 2a_1$ for a suitable positive integer a_1: substituting and replacing, we find... guess what?,

$$4a_1^3 + 2b_1^3 = c_1^3 \,,$$

which is a solution over the positive integers with $c_1 < c_0$. (Note that we need to have proved that $c_0 > 0$; otherwise, $c_1 = c_0/2$ could have been zero as well.) We have thus discovered that the smallest counterexample c_0 was not the smallest: then there was no c_0 in the first place, and the equation does not have a solution on the positive integers.

Problem 2.23.

If a sequence such as in (2) exists, then the set of its terms does not have a minimum: however given an element, there will be another element (for example, the next one in the sequence) which is strictly smaller. In this case, R has a subset which is not well ordered, so it is not well ordered.

If R is not well ordered, take a nonempty subset S of R which has no minimum. Choose $r_0 \in S$: as r_0 is not the minimum of S, there exists $r_1 \in S$ which is strictly smaller than r_0. Similarly, as r_1 is not the minimum of S, there exists $r_2 \in S$ which is strictly smaller than r_1. Iterating the procedure, we obtain a sequence of elements of R such as in (2). More in detail:

1. We choose the starting element $r_0 \in S$ as we want.

2. For every $n \in \mathbb{N}$, after we have chosen $r_n \in S$, we choose $r_{n+1} \in S$ so that it is smaller than r_n. This is always possible, because S has no minimum, so in particular r_n is not the minimum of S.

At the end of the exercise, note that R itself *can* have a minimum without being well ordered. For example, the set $\left\{ \dfrac{m}{n+1} \mid m, n \in \mathbb{N} \right\}$ has 0 as its minimum, but is not well ordered, because it contains the infinite decreasing sequence $1 > \dfrac{1}{2} > \dfrac{1}{3} > \cdots$.

Problem 3.2.

(a) In this case, R is verified, Q is not, and P is irrelevant: the assertion translates as R **and not**(Q).

(b) Here P, Q, and R are all verified, so this assertion translates as P **and** Q **and** R. Recall that **and** is associative, so $(P$ **and** $Q)$ **and** R is equivalent to P **and** $(Q$ **and** $R)$.

(c) Here we have a clear implication, and a causal one too! What the assertion says, is that if you get an A in the class, it means that you had gotten an A in the final: the translation in mathematical language is then R **implies** P.

(d) In this case, P is true, Q is false, and R is true: the assertion translates as P **and not**(Q) **and** R.

At the end of the exercise, observe how, in mathematical language, "but" and "nevertheless" mean the same as "and". The differences in the *tone* of the three words are lost in translation.

Problem 3.5.

(a) We first construct the truth table for P **and** Q **and** R, as it is almost immediate:

P	Q	R	P **and** Q **and** R
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**F**
T	**F**	**F**	**F**
F	**T**	**T**	**F**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

For the formula

$$S ::= (P \textbf{ implies } Q) \textbf{ and } (Q \textbf{ implies } R) \textbf{ and } (R \textbf{ implies } P)$$

we proceed in two steps: first, we construct the truth values for each of the implications; then, we compute those for their conjunction.

P	Q	R	P **implies** Q	Q **implies** R	R **implies** P	S
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**T**	**F**
T	**F**	**T**	**F**	**T**	**T**	**F**
T	**F**	**F**	**F**	**T**	**T**	**F**
F	**T**	**T**	**T**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**	**T**	**T**

We then see that, if P, Q and R are all **F**, then (3) is **T** and (4) is **F**.

Alternatively (as we did in classroom, for reasons of time) we could have observed that if P, Q, and R are all false, then P **implies** Q, Q **implies** R, and R **implies** P are all true, so S is true, while **F and F and F** is clearly false.

(b) Sam is silently assuming that *some* of P, Q and R are true. But why should it be so? All he has proved is that they are *equivalent:* either they all all true, or all false. To check which is the case, he must find a proof or disproof of any of the three (no matter which) which does *not* depend on the others, but only on other things which he *knows*, not just assumes, to be true.